STUDIES ON BEHAVIORS OF NONLINEAR DYNAMICAL SYSTEMS SUBJECTED TO RANDOM INPUTS

BY

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Abstract

In this dissertation, some classes of stochastic problems are studied for nonlinear dynamical systems subjected to a white Gaussian random noise.

The existence of stationary response and their behaviors are firstly considered for nonlinear systems subjected to random inputs. Two new approaches are developed to give sufficient conditions of the existence of the stationary probability density function for the response of nonlinear dynamical systems. The principal line of attack is directed to show existence conditions of an invariant measure related to the stationary probability density function. Furthermore, in order to explore stochastic behaviors of nonlinear dynamical systems of non-degenerate type, two theorems are demonstrated giving sufficient conditions for the existence of the stationary response and for the convergence of sample trajectories to the stationary state with a certain probability appraisal, based on the knowledge of sample properties of positive recurrent type.

Secondly, emphasizing the influence of the initial state on dynamic behaviors of a general class of nonlinear systems, a new approach to analyze the asymptotic behavior is developed. A new type of stochastic Lyapunov function which plays a key role to solve the problem is constructed, taking the dependence on the initial states into account. Several theorems are stated giving sufficient conditions of the asymptotic stability. The approach presented here is directly extended to a class of nonlinear stochastic

systems involving a random parameter modeled by a finite state Markov chain process.

Thirdly, we mainly discuss the noise stabilization of a general class of second order nonlinear dynamical systems. The theoretical method is the application of the averaging principle due to Khas'minskii as well as the properties of the singular points of Markov process generated by the Itô's nonlinear stochastic differential equation. By choosing the stabilizing noise term in an appropriate form, the singular point is obtained and sample path behaviors around the singular point are examined. Thus, the possibility for realizing the noise stabilization on Duffing-type nonlinear dynamical systems is theoretically concluded. Furthermore, based on the classification of the singular points, the general rules are established for realizing the noise stabilization of a general class of second order nonlinear dynamical systems.

Finally, a probabilistic approach is developed for the purpose of exploring the jump phenomena occurring in the response of a general class of nonlinear dynamical systems subjected to a narrow-band random input. The key notion is to derive the relation between probability density functions with respect to the squared values of the magnitudes of the response and the related narrow-band input. Through the variational averaging principle, the multi-valued response of the system is evaluated, including the theoretical examination of generating mechanism of jump phenomena.

CONTENTS

CHAPTER I INTRODUCTION

1.1 Introduction

Recent developments of dynamical and control system sciences have given rise to new problems in mechanical and structural vibrations and control system responses. The system response always fluctuates in a random manner and contains a wide spectrum of frequencies that may result in unwanted vibration in dynamical systems or structural ones. For example, measurements of a ship motion on the sea or an aircraft flying through turburent air reveal that such motions may be described only by the stochastic method. Earthquakes are also examples of random processes that can excite severe vibration and failure in buildings. Many physical systems encountered in the fields of aeronautical and ocean engineering and the response of structures to earthquakes, etc. show the following two aspects in common; (a) they involve a system response to random excitation; (b) in general, they exhibit various kinds of nonlinear behaviors such as limit cycles, jump phenomena, etc., because almost all real systems exhibit nonlinear characteristics. Behaviors of nonlinear systems under random excitation are of considerable importance to those engaged in studies of system analysis and design in structural fatigue and control engineering.

1.2 Scope of Problems

This thesis consists of some important problems of interests in the response of nonlinear dynamical systems subjected to a random excitation.

[Problem 1] Existence of Stationary Responses and Their Behaviors

In practical problems related to nonlinear stochastic systems, an important topic is the analysis of the steady state behavior of the system. This is the case in which, as time goes on, the transition probability density function or the conditional probability density function tends to a stationary probability density function. For the analysis of steady state behaviors, generally, the Fokker-Planck equation for the transition probability density can be used, which plays an important role to explore the behavior of Markov processes. However, exact solutions are rarely known except for simple systems under white Gaussian noise excitation. The purpose of the reseach is directed to obtaining the condition for the existence of the stationary probability density function. New approaches to the analysis of nonlinear stochastic systems are developed, based on the concept of an invariant measure related to the stationary probability density function.

[Problem 2] Asymptotic Stability

The problem of great importance is the asymptotic behavior of nonlinear stochastic systems dependent on their initial states, which shows an inherent characteristics due to the existence of nonlinearities. As the procedure for examining the system stability, the stochastic Lyapunov function approach has become well known. A difficult step in the application of Lyapunov theory to analyse the stability of stochastic systems is the construction of a suitable Lyapunov function. There is indeed no general systematic procedure for generating Lyapunov functions. This thesis is concerned with developing a realizable approach to solve stochastic asymptotic stability for nonlinear systems, by constructing the stochastic Lyapunov function taking into account the effect of the initial states, associated with (1) a random parameter modelled by white Gaussian random process and (2) two random parameters modelled by a white Gaussian and a finite state Markov chain processes respectively.

[Problem 3] Noise Stabilization

The problem of stabilization of nonlinear dynamical systems through the introduction of noise sources appears to be of great practical significance. It is well known that unstable systems can be stabilized by the introduction of a deterministic signal, in particular, a sinusoidal one of sufficiently high frequency[36]. It has also been observed that deterministic systems operating in a random environment possess stable characteristics and yet, when the randomness is taken out of the environment, the system becomes unstable. From the viewpoint mentioned above, our final goal is to clarify the situation in order to allow us to predict when an arbitrary random environment will stabilize an otherwise unstable state of a nonlinear system, or under what conditions an unstable state will be made stable by the introduction of noise.

[Problem 4] Jump Phenomenon

In nonlinear dynamical or control systems, it is well known that the jump resonance may often occur with a consequent worsening of control performances. Hence, in analysis and synthesis of non-linear dynamical or control systems, it is very important to find whether jump resonance can occur or not.

In deterministic systems, the jump phenomenon implies that the amplitude and phase angle of the output exhibit jump response, as the amplitude or frequency of the input is varied. On the other hand, for systems with random inputs, in the past publications, the jump phenomenon occurs between the input and the output variances of the system response and the theoretical approach to analize the jump phenomenon was based on the statistical linearization method to stationary random input. However, the approach described above can not give enough explanation of the response when jump phenomenon occurs, because it becomes difficult to comprehend stochastically sample path behaviors of the response. The main purpose of the study is to clarify the generating mechanism of jump resonance occurring in nonlinear dynamical or control systems by evaluating the probability density functions of input and output, in which sample path behaviors will be shown with a digital simulation experiment. Here, it is assumed that the random input to nonlinear systems is a narrow-band random process whose signal power is restricted to a very narrow frequency range. The results are compared with the jump phenomenon in nonlinear dynamical systems subjected to a sinusoidal input which has been already investigated in detail.

1.3 Historical Background

For convenience of present descriptions, the historical background is separately retrospected into the four versions.

1.3.A On the Stationary Response of Nonlinear Stochastic Systems

The investigation of evaluating the response of dynamical or control systems subjected to random inputs was first developed by the statistical theory. Many studies have appeared on responses of linear systems with random noise by Laning & Batten[1], Crandoll[2], Davenport & Root[3] and Lee[4], et al. On the other hand, in the case of nonlinear systems, such approximate methods as the perturbation and the statistical linearization have been developed in order to extend a linear method of analysis to certain systems containing small nonlinearities by Booton[5], Caughy[6], Pervonzvanskii [7] and Sawaragi, Sugai and Sunahara[8]. These studies were made by computing various response statistics such as mean-square response, correlation function and response spectral density.

An area where more results are expected is the Fokker-Planck equation associated with nonlinear stochastic systems. Although there are a few papers with respect to the analysis of Fokker-Planck equation by Fuller[9], Stratonovich[10], et al., it is difficult to solve directly the Fokker-Planck equation for nonlinear systems. The analysis of stationary responses of nonlinear dynamical systems is to explore the existence of the stationary probability density function of the Fokker-Planck equation. Recently, as a useful analytical method for the existence of stationary responses, the concept of an invariant probability measure[11],[12],[13] was introduced which was related to the stationary probability density function. Using the concept of an invariant measure, necessary and sufficient conditions for the existence of a unique invariant measure were first given by Khas'minskii[11] with respect to recurrent diffusion processes. Existence and uniqueness conditions of an invariant measure of the solution process to a scalar stochastic differential equation of Itô-type were shown by Itô and Nisio[14]. Following the results by Khas'minskii, Wonham[15],[16] established existence conditions of an invariant measure of vector stochastic differential equations of Itô-type, where the solution process was restricted to the strongly Feller process. An extensive study was reported by Beneš[12] and Foguel[13] on the existence of an invariant measure of Markov processes. Following the results of Beneš and the mean ergodic theorem, Zakai[17] established the condition for the existence of an invariant probability measure for Feller process. From the practical point of view, Sunahara, the Author and Morita[18] has developed two new approaches to give sufficient conditions of the existence of the stationary probability density function for the response of nonlinear dynamical systems, based on the concept of an invariant measure.

The stochastic stability[19] has been studied by many investigators. Pinsky[20] has given various conditions for the asymptotic stability of the origin for a linear stochastic differential equation in both degenerate and non-degenerate cases, with a slight different concept of the stochastic Lyapunov stability. For nonlinear stochastic systems in the non-degenerate case, Wonham[16] defined the weak stochastic stability corresponding to Lagrange stability[21] in the deterministic system, based on the concept of positive recurrent for the diffusion process. He showed that sufficient conditions for recurrence and positivity were given through the existence condition of an invariant measure for the diffusion process defined by the stochastic differential equation of Itô-type. Itô & Nisio[14] showed that the conditions for the diffusion process to be a positive recurrent type can be characterized in terms of Feller's probability measures[22],[23] and developed some general properties of the diffusion process of non-degenerate type in the non-singular intervals on which Feller's probability measures are given. Sunahara and the Author[24] has established a new analytical method for exploring stochastic behaviors of nonlinear dynamical systems of non-degenerate type, based on the knowledge of sample properties of the diffusion process of positive recurrent type.

1.3.B <u>On the Asymptotic Stability of Nonlinear Stochastic Systems</u>

In this section, we will survey studies on stochastic stability problems for systems governed by continuous time Markov processes. Our concern will mainly be with the asymptotic behavior of sample processes. Gihman & Skorohod[25] considered the asymptotic stability of solutions in the mean of second-order moments of linear stochastic differential equations and their asymptotic behaviors. Khas'minskii[26] gave necessary and sufficient conditions for stability in probability of an equiliblium solution to a class of linear stochastic differential equation of Itô-type. Based on Khas'minskii's theory, Kozin[27],[28] established some theorems concerning necessary and sufficient conditions for almost sure sample stability of second-order linear stochastic systems.

On the other hand, for the stability analysis of nonlinear stochastic systems, the most useful technique is an extention of the deterministic Lyapunov theory[21] to nonlinear stochastic systems. Bucy[29] recognized that stochastic Lyapunov functions should have the super-martingale property and proved a theorem on "with probability one" convergence for discrete parameter processes. Bucy's work is probably the first one to treat a nonlinear stochastic stability problem by the extension of deterministic Lyapunov theory. Some results, of the Lyapunov form, were given by Khas'minskii[30]. Kushner[31],[32] extended the idea of Bucy to the continuous parameter systems and thus the range of applicability of the stochastic Lyapunov function results of Khas'minskii. Also, Wonham[16] derived a weaker sufficient condition than Khas'minskii's sufficient condition of the stochastic stability. Furthermore, for the construction of a suitable Lyapunov function, Kushner[33] proposed a method for construction of stochastic Lyapunov functions. The stochastic stability theory untill now was mainly developed only in the ε-neighborhood of an equiliblium point. However, the asymptotic stability of nonlinear stochastic systems depends strongly on the initial conditions. Taking into account the influence of initial conditions to stochastic stability, Sunahara, the Author & Morita[34] have developed a new stochastic Lyapunov function approach to explore the asymptotic stability for a general class of nonlinear dynamical systems with a random parameter modelled by a white Gaussian random process.

In the case of nonlinear dynamical systems with a random parameter involving a finite state Markov chain process, the concept of random evolutions by Griego & Hersh[35],[36], Hersh & Papanicolau [37] and Hersh & Pinsky[38] is introduced instead of the averaging principle because of the existence of parameters of a Markov chain process. Based on the concept of random evolutions, Sunahara, the

Author & Morita[39] investigated asymptotic stability of nonlinear dynamical systems with two kinds of random parameters modelled by a white Gaussian and Markov chain process respectively.

1.3.C <u>Stabilization of Nonlinear Dynamical Systems</u>

In this section, we shall briefly discuss some results on stabilization of nonlinear dynamical systems as a rather significant problem than the stochastic stability theory. A study of the stabilization of unstable dynamical or control systems originated in 1956 by Ordenburger[40]. Oldenburger[40] has discovered that the amplitude of the sustained oscillation which can be observed in an unstable nonlinear control system either decreases or disappears by applying a sinusoidal signal with the high frequency and the sufficiently small amplitude. Lowenstern[41] has given suggestions for the stabilization of unstable dynamical systems through the statistical analysis of oscillations in a parametrically excited linear dynamical system for a restricted class of random excitations. A theoretic ascertion of the stabilization of nonlinear control system has already been established by Sawaragi, Sugai & Sunahara[8] and Oldenburger & Sridhar[42] through the use of the statistical linearization technique. The statistical linearization technique is essentially a stochastic counterpart of the describing function method which is commonly used for studying the characteristics of nonlinear deterministic systems.

Furthermore, Bogdanoff & Citron[43] has reported on experimental results of stabilizing an inverted pendulum with vertical, almost periodic base motion. Theoretical conclusions that verify the experimental results were investigated through the use of averaging method established by Bogoliubov[44] for systems with almost periodic time-varying parameters, in order to achieve approximate moment stability results. Afterwards, the principle of averaging was extended to parabolic and elliptic differential equations and to Markov processes with a small diffusion by Gikhman[45], Khas'minskii[46] and Mitropol'skii & Kolomiets[47]. It was clear that the method of averaging as applied to stochastic systems was related to [43]. Using the extended averaging method[46], Mitchell [48] has studied noise signals of an almost periodic type in connection with stabilization of an inverted pendulum. Binia, et al. [49] has treated the problem associated with nonlinear oscillators driven by noise and Samuels[50] associated with the stabilization of deterministic, linear, unstable RLC circuit by the introduction of a white Gaussian noise to system parameters.

From theoretical viewpoints which we should examine sample path behaviors instead of moment properties, Sunahara, Kozin & the Author[51],[52] have shown the possibility of noise stabilization for unstable nonlinear dynamical systems by applying the extended averaging principle[46] and, furthermore, has established a general rule for realizing the noise stabilizatiin of a general class of second-order nonlinear systems.

1.3.D On Jump Phenomenon of Nonlinear Dynamical Systems

In the case of periodic input signals, using the describing function method, rigorous conditions for generating jump resonance have been completely studied by Sandberg[53], Hatanaka[54] and Hayashi[55], et al. On the other hand, for systems with random inputs, the method of statistical linearization due to Booton[56], et al. has widely been used for the reseach of jump phenomenon. Kyong[57] presented the statistical linearization criteria for unique response for several common nonlinearities and showed that an idealized saturation and an idealized deadzone yield jump phenomenon among a restricted class of nonlinearities. Sawaragi & Sunahara[58] recognized the jump phenomenon in the relation curves between the standard deviation of stationary random input and that of error signal, using the equivalent linearization technique and also verified the validity of the above theoretical investigation through experimental studies of an analog computer.

Lyon, et al.[59] demonstrated analytically and experimentally that jumps can occur when the oscillator is subjected to a narrow-band random noise. The analytical work of Lyon, et al. is based on a linearization method for which the necessary condition is that the magnitude of fluctuations must be restricted. Using the associated Fokker-Planck equation, Sunahara & the Author[60] has developed a probabilistic approach to explore the generating mechanism of the jump phenomenon occurring in a general class of nonlinear dynamical systems subjected to a narrow-band random input.

1.4 Summary of Contents

In this dissertation, some classes of stochastic problems of nonlinear dynamical systems subjected to white Gaussian random noise are studied, i.e., (1) Existence of stationary responses and their behaviors, (2) Asymptotic stability, (3) Noise stabilization and (4) Jump phenomenon.

Chapter 2 is devoted to mathematical preliminaries related to the theory of stochastic processes which will be used in the succeeding developments. The mathematical model of the system is established by the theory of Itô-type stochastic differential equations.

In Chapter 3, two new approaches are developed to give sufficient conditions for the existence of the stationary probability density function for the response of nonlinear dynamical systems. The principal line of attack is directed to show existence conditions of an invariant measure related to the stationary probability density function. Two approaches are presented : one is to choose a suitable Lyapunov-like function and another to find out an arbitrary function satisfying the martingale property of Markov processes. A new analytical approach is developed in Chapter 4 to explore stochastic behaviors of nonlinear dynamical systems of non-degenerate type. The key problem is to examine the existence of an invariant measure for stochastic systems with the differential generator of non-degenerate points. Two theorems are demonstrated giving sufficient conditions for the existence of the stationary response and for the convergence of sample trajectories to the stationary state with a certain probability appraisal, based on the knowledge of sample properties of positive recurrent type.

In Chapter 5, emphasizing the influence of the initial state on dynamic behaviors of a general class of nonlinear stochastic systems, a new approach to analize the asymptotic behavior is developed, where a new type of stochastic Lyapunov function plays a key role to solve the problem, taking the dependence on the initial states into account. The mathematical model of a dynamical system is given in the form of a general class of nonlinear differential equations with a state dependent random parameter. Several theorems are stated giving sufficient conditions of the asymptotic stability in the case where the random parameter is modeled by a white Gaussian noise process multiplied by a nonlinear function. Furthermore, the approach presented here is directly extended to a class of nonlinear stochastic systems with a random parameter modeled by a finite state Markov chain, using the concept of random evolutions.

Chapter 6 develops the noise stabilization of a class of second-order nonlinear dynamical systems. The analytical method is based on the application of the averaging principle established by Khas'minskii. The noise stabilization term added to the system is selected in the modified form of the white Gaussian noise process. The determination of a stabilizing signal can be performed through the procedure that the singular point at where the diffusion disappears is obtained and sample path behaviors around the singular point are examined. Thus the possibility for realizing the noise stabilization on Duffing type nonlinear dynamical systems is theoretically shown. Chapter 7 is concerned with extensions of the method in Chapter 6 to a general class of nonlinear dynamical systems. In this Chapter, general conditions are obtained through an application of the averaging principle due to Khas'minskii as well as the properties of the singular points of Markov process generated by the Itô-type nonlinear differential equation. The classification of singular points is established in terms of relative relations of both the drift and diffusion terms. Applying the general rules established here, stabilization studies are performed in a number of classical cases for various noise coefficients.

In Chapter 8, a probabilistic approach is developed for the purpose of exploring the jump phenomenon occurring in the response of a general class of nonlinear dynamical systems subjected to a narrow-band random input. The response of nonlinear dynamical systems considered is related to the narrow-band input generated as the output of a lightly damped linear system excited by a white Gaussian signal. The relation is derived between probability density functions with respect to the squared values of magnitudes of the response and the related narrow-band input. The multi-valued response of the system is evaluated, including the theoretical examination of generating mechanism of jump phenomenon.

Throughout all chapters, digital simulation studies are demonstrated to show the validity of the theories presented.

Chapter 2 Mathematical Preliminaries

2.1 Diffusion Process

2.1.A Definition of Diffusion Process

Let (E,B) be an arbitrary measurable space, E a finite or countable set and B the σ-algebra of the measurable sets generated by the open sets of the space (E,B). Let us consider a given Markov transition function $P(t,x,A)$, $x \in E$, $A \in B$, $t \geq 0$. Together with the fundamental properties of the function $P(t,x,A)$[61], we shall assume that $P(t,x,E)=1$ for all $t \geq 0$. That is, the corresponding Markov process is not cut off. Furthermore, the following conditions (C-1)~(C-3) will be assumed to be fulfilled.

(C-1) For any arbitrary ε-neighborhood $U_\varepsilon(x)$ of the point x, $1-P(t,x,U_\varepsilon(x)) = o(t)$ uniformly in x in an arbitrary compactum $K \subset U$.

(C-2) For an arbitrary bounded B-measurable function f and for each $t>0$, $T_t f(x) = \int_E P(t,x,dy)f(y)$ defines a continuous function of x,*1 where T_t is the semi-group of linear operators.

(C-3) For an arbitrary $x \in E$, $t>0$, and for any open set U, the relation $P(t,x,U) > 0$ holds.

Then, it is known[61] that there exists a homogeneous Markov process $x(t)$ satisfying

(C-4) For any compactum K and any $s>0$,
$P_x[\{x(t) \in K, 0 \leq t \leq s\} \cap \{x(t) \text{ has a discontinuity for } 0 \leq t \leq s\}] = 0$.

(C-5) $x(t)$ is a strong Markov process.

A process $x(t)$ for which conditions (C-1)~(C-5) are satisfied is called a diffusion process.

2.1.B The Kolmogorov Equations for Diffusion Processes *2

Let $\{x(t), t \geq 0\}$ be a continuous stochastic process of the Markov type defined on the real line ; that is, $x(t)$ is a Markovian random variable, depending on a continuous parameter t, which assumes values in the state space $R = \{x; -\infty \leq x \leq \infty\}$. In this section, we derive and study the Kolmogorov diffusion equations associated with continuous Markov processes on the real line. Let

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*1 This property has been studied in detail by Girsanov in [62]. Processes for which (C-2) is satisfied are called strongly Feller in [62] .

*2 For studies on Kolmogorov equations for diffusion processes, we refer to the books of Fuller[9], Dynkin[61], Friedman[63], Itô [64] and Bharucha-Reid[65], e.t.c.

$$(2.1) \qquad P(t,x;\tau,y) = P_r\{x(\tau)<y \mid x(t)=x\}, \qquad \tau>t$$

denote the transition probabilities of the process $\{x(t),t\geq 0\}$. Naturally, it can be also written that

$$(2.2) \qquad P(t,x;\tau,y) = P(\tau-t,x,y).$$

For t and x fixed, $P(t,x;\tau,y)$ is a continuous function of τ. In addition, $P(t,x;\tau,y)$ is a (conditional) distribution function in y satisfying the usual conditions,

$$(2.3) \qquad \lim_{y\to-\infty} P(t,x;\tau,y) = 0, \qquad \lim_{y\to\infty} P(t,x;\tau,y) = 1.$$

If the mean and variance of the change in x(t) during the time interval Δt are defined by the following truncated moments,

$$(2.4) \qquad \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|y-x|<\delta} (y-x)d_y P(t-\Delta t,x;t,y) = b(t,x)$$

and

$$(2.5) \qquad \lim_{\Delta t\to 0} \frac{1}{\Delta t} \int_{|y-x|<\delta} (y-x)^2 d_y P(t-\Delta t,x;t,y) = a(t,x),$$

the backward Kolmogorov equation is obtained by [60]

$$(2.6) \qquad -\frac{\partial P(t,x;\tau,y)}{\partial t} = \frac{1}{2}a(t,x)\frac{\partial^2 P(t,x;\tau,y)}{\partial x^2} + b(t,x)\frac{\partial P(t,x;\tau,y)}{\partial x}.$$

Similarly, the density function $p(t,x;\tau,y)$ satisfies

$$(2.7) \qquad -\frac{\partial p(t,x;\tau,y)}{\partial t} = \frac{1}{2}a(t,x)\frac{\partial^2 p(t,x;\tau,y)}{\partial x^2} + b(t,x)\frac{\partial p(t,x;\tau,y)}{\partial x}.$$

We can also derive the so-called Kolmogorov's forward equation, which is also called the Fokker-Planck equation, as given by

$$(2.8) \qquad \frac{\partial p(t,x;\tau,y)}{\partial t} = \frac{1}{2}\frac{\partial^2[a(\tau,y)p(t,x;\tau,y)]}{\partial y^2} + \frac{\partial[b(\tau,y)p(t,x;\tau,y)]}{\partial y}.$$

The forward equation is the formal adjoint of the backward equation, and it will be of interest in this thesis.

Equations (2.7) and (2.8) are rewritten respectively by

$$(2.9) \qquad -\frac{\partial p}{\partial t} = Lp$$

where

$$(2.10) \qquad L(\cdot) = \frac{1}{2}a(t,x)\frac{\partial^2(\cdot)}{\partial x^2} + b(t,x)\frac{\partial(\cdot)}{\partial x}$$

and

$$(2.11) \qquad \frac{\partial p}{\partial \tau} = L^* p$$

where

$$(2.12) \qquad L^*(\cdot) = \frac{1}{2}\frac{\partial^2[a(\tau,y)(\cdot)]}{\partial y^2} + \frac{\partial[b(\tau,y)(\cdot)]}{\partial y}.$$

The operator L is called the differential generator of the diffusion process and the quantities $a(t,x)$, $b(t,x)$ are called respectively the diffusion coefficient and the drift coefficient. Also, L^* is the conjugate differential operator of L.

Let both $a(t,x)$ and $b(t,x)$ in Eq.(2.10) depend on x, and not on t. For the term $a(x)$ in (2.10), if there exists a point $x=x_s$ satisfying $a(x)=0$, the differential generator L of (2.10) is said to be <u>degenerate type</u>. Otherwise, if $a(x) \neq 0$ for all x, (2.10) is said to be <u>non-degenerate type</u>.

2.1.C Diffusion Process on the Real Line

Let $x(t,\omega)$ be a Markov process in $E^{(1)}$ starting at the point ζ in the half-infinite interval $[0,\infty)$ with the differential generator,

$$(2.13) \qquad L = U^2(x)\frac{\partial^2}{\partial x^2} + V(x)\frac{\partial}{\partial x}$$

where both the drift coefficient $V(x)$ and the diffusion coefficient $U^2(x)$ are polynomials in x and these satisfy Lipschitz and uniform growth conditions. Since diffusion processes are defined to be Markov processes with continuous trajectories, this leads in a natural way to a discussion of x(t) as a diffusion process on the interval $I=[r_1,r_2]$ where $-\infty \leq r_1 \leq r_2 < \infty$.

Interesting situations arise when the diffusion is singular for which the following relation holds,

(2.14) $U^2(x) = 0.$

A point $x=r_s$ for which $U^2(r_s)=0$ is called the singular point, where $r_1 \leq r_s \leq r_2$. There are two types of singular points depending on the value of the drift coefficient V(x) at the singularity.[66]

[Definition 2.1] A point r_s for which $U^2(r_s)=0$ is called a trap, provided that $V(r_s)=0$.

[Definition 2.2] A point r_s for which $U^2(r_s)=0$ is called a right (left) shunt, provided that $V(r_s)>0$ (<0).

From the physical viewpoint, it may be observed that a sample process of the diffusion process is obviously singular at the point $x=r_s$, because, if there is no diffusion there, the process becomes

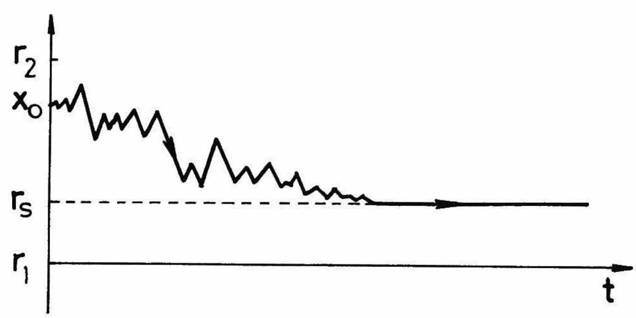


Fig.2.1 A sample process x(t) in the case where $U^2(r_s)=0$.

deterministic at that point as shown in Fig.2.1. It is also obvious that a sample process does not move any more at the point $x=r_s$, provided that there are no diffusion and no drift at that point. This situation expresses the trap. A positive (negative) drift at the singularity causes the particle to be shunted to the right (left). These heuristic discussions are stated in probabilistic terms by the following lemma.

[Lemma 2.1] [67],[68]

Let $P_r\{\cdot\}$ be the probability of the event "$\cdot$" associated with the trajectory $x(t)$ with the initial value $x(0)=x_0$.

(1) If a point r_s is a trap, then

(2.15) $\quad P_r\{\ x(t)=r_s \text{ for all } t\geq 0\ \} = 1.$

In this case, almost all trajectories which originate at ζ remain at that point. Also, almost all trajectories which originate to one side of the trap will never cross it. Either the trap is never reached or if it is reached, the trajectory stays there.

(2) If a point r_s is a left shunt, then

(2.16) $\quad P_r\{\ x(t)<r_s \text{ for all } t\geq 0\ \} = 1.$

In this case, almost all trajectories originating at a point r_0 instantaneously leave that point to enter the neighborhood to the left side of a point r_s. Therefore, a trajectory never returns to r_s with probability one. A similar statement holds to a right shunt.

Lemma 2.1 gives a considerable amount of knowledges about sample trajectories at or near singularities. However, this is not sufficient to determine the stability of sample processes. In order to make the sample stability more clear, we shall classify the boundaries of the interval $I=[r_1,r_2]$ formed by the singularities on the r-direction. Detailed aspects of the classification may be found in References[22],[23],[68] to the present case. For the classification of boundaries, we shall consider the differential generator (2.13) on the interval $[r_1,r_2]$ and define the function,

$$(2.17)\qquad B(x) = \int_{r_0}^{x} 2V(\xi)U^{-2}(\xi)d\xi$$

where r_0 is a fixed value in I. Also, the canonical scale and the canonical measure are introduced, which are respectively defined by

$$(2.18)\qquad ds(x) = \exp\{-B(x)\}dx$$

$$(2.19)\qquad dm(x) = 2U^{-2}(x)\cdot\exp\{B(x)\}dx,$$

where s(x) is a continuous to the right and increasing function on $[r_1,r_2]$ and m(x) a continuous and increasing function. We shall denote,

$$(2.20a)\qquad \sigma_1 = \iint_{r_1<x<y<r_1'} dm(x)ds(y),$$

$$(2.20b)\qquad \mu_1 = \iint_{r_1<x<y<r_1'} ds(x)dm(y),$$

$$(2.20c)\qquad \sigma_2 = \iint_{r_2>y>x>r_2'} dm(x)ds(y),$$

$$(2.20d)\qquad \mu_2 = \iint_{r_2>y>x>r_2'} ds(x)dm(y).$$

The boundaries of the interval $[r_1,r_2]$ are classified according to the behaviors of the speed and scale measures near r_1 and r_2 via the functions σ_i and μ_i (i=1,2).

The boundaries are first classified as to whether they are accessible or inaccessible and then further subdivided into regular or exit if accessible and entrance or natural if inaccessible.

[Definition 2.3] The boundaries r_i, $i=1,2$ are classified as follows:

$$r_i \text{ is accessible if } \sigma_i<\infty \quad \begin{cases} \text{regular} & \text{if } \mu_i<\infty \\ \text{exit} & \text{if } \mu_i=\infty \end{cases}$$

$$r_i \text{ is inaccessible if } \sigma_i=\infty \quad \begin{cases} \text{entrance} & \text{if } \mu_i<\infty \\ \text{natural} & \text{if } \mu_i=\infty. \end{cases}$$

A boundary is accessible if there is some probability that it will be reached in a finite time [68]. Otherwise it is inaccessible. However, the explanation of the inaccessibility is not sufficient for the behavior of the process in the interval $[r_1,r_2]$ because, as was discovered by Doob [68], a natural boundary can be asymptotically approached with probability one although it is never reached. This leads to a further subdivision of inaccessible (natural) boundaries.

[Definition 2.4] An inaccessible (natural) boundary r_i will be called locally attractive if $s(r_i)$ is finite and locally unattractive if $s(r_i) = \pm\infty$.

The asymptotic behaviors of x(t) trajectories in the interval $[r_1,r_2]$ can now be determined in terms of these difinitions 2.1, 2.2,2.3 and 2.4. As examples: (1) $\{r_1,r_2\}$ = {exit, entrance}. As shown in Case 1 in Fig.2.2, the probability is zero that the x(t)-process can reach the boundary r_2 and almost all trajectories leave the interval $[r_1,r_2]$ at r_1. The boundary r_1 is either 'trap' or 'absorbing boundary' and the boundary conditions must be imposed.

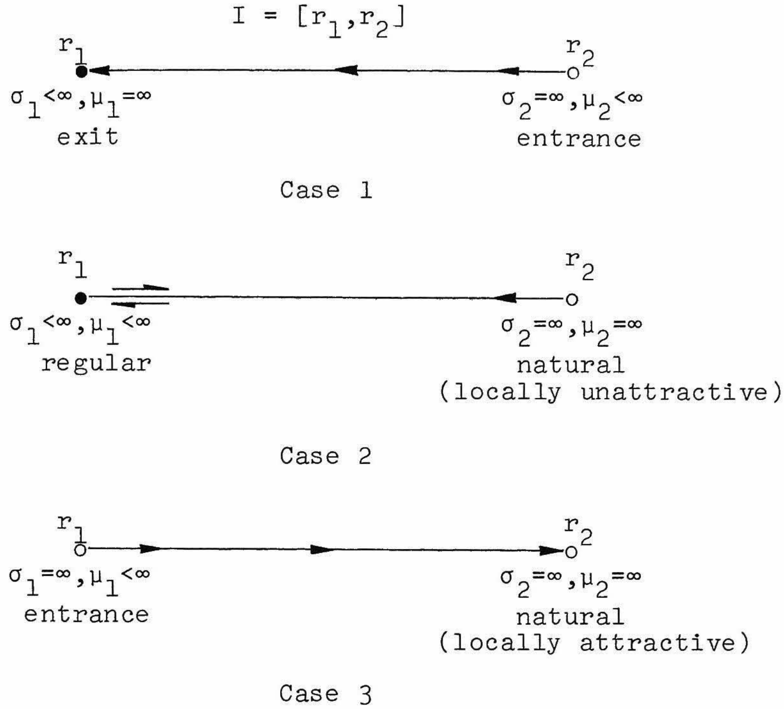


Fig.2.2 Illustrative Examples of the x(t)-trajectories by Classification of Boundaries.

(2) $\{r_1, r_2\}$ = {regular, natural (locally unattractive)}. As shown in Case 2 in Fig.2.2, the process can reach the boundary r_1 with some probability in a finite time and the behavior after reaching r_1 can be determined by imposing boundary conditions. Also, the process can not go to the boundary r_2 because r_2 is locally unattractive natural boundary. (3) $\{r_1, r_2\}$ = {entrance, natural (locally attractive)}. As shown in Case 3 in Fig.2.2, almost all trajectories originating within the interval $[r_1, r_2]$ approach the boundary r_2 asymptotically as $t \to \infty$ without ever reaching r_2, since r_2 is locally attractive natural boundary.

2.2 Averaging Principle

The averaging principle has been established for partial differential equations of the form, $\partial u/\partial t = \varepsilon L(t,x)u$, where L is the second-order elliptic or parabolic differential operator and ε sufficiently small constant.

The averaging principle is stated as follows,

[Theorem] (Khas'minskii)[46]

The solution of the Cauchy problem for the partial differential equation of the form $\partial u(t,x)/\partial t = \varepsilon L(t,x)u(t,x)$ as $\varepsilon \to 0$ may uniformly be approximated over an interval of time which is $O(1/\varepsilon)$ by the solution of the equation $\partial v(\tau,x)/\partial \tau = \varepsilon L^0(x)v(\tau,x)$ where τ is $O(t/\varepsilon)$ and L^0 is an operator whose coefficients are obtained from those of $L(t,x)$ by averaging with respect to time, where $L^0(x)$ is described by

$$(2.21) \qquad L^0(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T L(t,x)dt.$$

More concisely, the following relation holds:

$$(2.22) \qquad \lim_{\varepsilon\to 0} P_r\{ \sup_{[t,x]\in E^{(1)}\times I_T} |u(t,x)-v(\tfrac{t}{\varepsilon},x)| = 0\} = 1$$

where $I_T = [0,T)$.

2.3 Symbolic Conventions

Principal symbols used here are listed below:

- t : Time variable, particularly present time
- t_0 : The initial time at where the system trajectory starts
- $x(t)$: A scalar or vector stochastic process representing

the system state

$f(t,x), g(t,x)$: A scalar or vector nonlinear function with respect to x

$\dot{\xi}(t)$: A white Gaussian random process

$w(t)$: A one-dimensional standard Brownian motion process

$E\{\cdot\}$: The mathematical expectation

$\mu(r)$: An invariant measure

$s(r)$: A canonical scale measure

$m(r)$: A canonical speed measure

$L(\cdot)$: The differential generator

$V_L(\cdot)$: A stochastic Lyapunov function

$P(t,x;\tau,\Gamma)$: The transition probability which the $x(\tau)$-process with $x(t)=x$ is included within Borel sets Γ, i.e., $P_r\{x(\tau)\in\Gamma|x(t)=x\}$

$p(t,x)$: The joint probability density function with respect to t and x

$\alpha(t)$: A Markov chain process

$z(t)$: A narrow-band random process

$E^{(n)}$: An n-dimensional Euclidean space

Chapter 3

Existence of Stationary Response for Nonlinear Dynamical Systems of Degenerate Type

3.1 Introduction

In recent years, considerable interests have arisen in the response of nonlinear dynamical systems subjected to random excitation. Among many practical problems related to nonlinear stochastic systems, extensive researches have been directed toward finding the existence of the stationary response for randomly excited nonlinear second-order systems. In many cases, exact solutions are not available and then methods of approximate analysis must be well developed for nonlinear systems. In spite of mathematical difficulties of nonlinear system analyses, the analysis of stationary responses has been developed through the evaluation of mean-square responses by utilizing such approximate methods as the perturbation and the statistical linearization and furthermore by measurements of autocorrelation function or spectral density.[1]~[5],[8]

On the other hand, another useful technique for exploring the stationary response of nonlinear stochastic systems is the application of the Fokker-Planck equation with respect to the probability density function of the solution processes. The exposition of the Fokker-Planck equation will be given, with emphasis on steady state solutions.

Let $\mathbf{r}(t)$ be the n-dimensional random process whose components are denoted by $x_1, x_2, \cdots, x_n$. We may now define the Markov process to mean that the conditional probability density function that $\mathbf{r}$ lies in the interval from $\mathbf{r}_1$ to $\mathbf{r}_1 + d\mathbf{r}_1$ at time t_1, from $\mathbf{r}_2$ to $\mathbf{r}_2 + d\mathbf{r}_2$ at time $t_2, \cdots$, from $\mathbf{r}_{n-1}$ to $\mathbf{r}_{n-1} + d\mathbf{r}_{n-1}$ at t_{n-1}, depends only on sample values of $\mathbf{r}$ at t_n and t_{n-1}, i.e.,

$$(3.1) \qquad p(\mathbf{r}_n, t_n | \mathbf{r}_1, t_1; \mathbf{r}_2, t_2; \cdots; \mathbf{r}_{n-1}, t_{n-1}) = p(\mathbf{r}_n, t_n | \mathbf{r}_{n-1}, t_{n-1}).$$

We shall write a general expression of the conditional probability density function by using a form of the transition probability density function $p(\mathbf{r}, t; \mathbf{r}(t+\Delta t), t+\Delta t)$. The transition probability density function $p(\mathbf{r}, t; \mathbf{r}(t+\Delta t), t+\Delta t)$ means a sample movement from $\mathbf{r}$ to $\mathbf{r}(t+\Delta t)$ during a time interval Δt, based on the assumption that the sample value was $\mathbf{r}$ at time t.

With background knowledge of the Markov process theory, a parabolic partial differential equation can be derived in the form,

$$(3.2) \qquad -\frac{\partial p}{\partial t} = \sum_i V_i \frac{\partial p}{\partial x_i} + \frac{1}{2} \sum_{i,j} U_{i,j} \frac{\partial^2 p}{\partial x_i \partial x_j}$$

where $i,j = 1, 2, \cdots, n$. In deriving Eq.(3.2), the following assumptions must be made. The first and second incremental stochastic moments of the movement in an infinitesimal period of time are proportional to Δt so that the following limits exist:

$$(3.3a) \qquad V_i = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{E^{(n)}} (\Delta x)_i p \, d(\Delta x)$$

and

$$(3.3b)\quad U_{ij} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{E^{(n)}} [(\Delta x)(\Delta x)']_{ij} p\, d(\Delta x)$$

where $[(\Delta x)(\Delta x)']_{ij}$ is the (i,j)-th element of $(\Delta x)(\Delta x)'$. A further assumption has been introduced in which the higher moments are of the order of (Δt).

Our problem is to examine whether or not there exists stationary solution $p^*(r)$ of Eq.(3.2) as $t \to \infty$. If the stationary probability density function $p^*(r)$ exists, this implies the existence of stationary responses in nonlinear stochastic systems. Then, by letting $t \to \infty$ and setting $\partial p/\partial t = 0$ in Eq.(3.2), the probability density function $p^*(r)$ may be obtained from the Fokker-Planck equation. However, it is generally difficult to find out the existence of the stationary probability density function $p^*(r)$ for nonlinear stochastic systems.

Now, we shall consider the one-dimensional r(t)-process with the following differential generator,

$$(3.4)\quad L_r(\cdot) = U^2(r)\frac{d^2(\cdot)}{dr^2} + V(r)\frac{d(\cdot)}{dr}.$$

This differential generator L_r plays an important role to analize the stationary responses, that is, the stationary probability density function $p^*(r)$, of nonlinear dynamical systems. Both the coefficients $U^2(r)$ and $V(r)$ in (3.4) imply that, in the representation of Itô-type stochastic differential equation, the former corresponds to diffusion based on the stochastic movement and the latter to drift based on the deterministic one respectively. From the description of Section 2.1.B, for the diffusion term $U^2(r)$ in

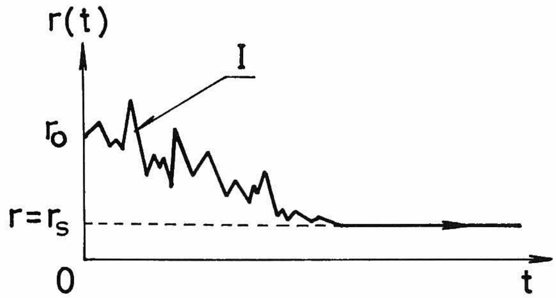


Fig.3.1 System Behavior of Degenerate Type

(3.4), if there exists a singular point $r=r_s$ satisfying $U^2(r)=0$, the dynamical system is said to be of degenerate type. Otherwise, if $U^2(r)\neq 0$ for all r, the dynamical system is said to be of non-degenerate type.

In this Chapter, we shall consider behaviors of the r(t)-process only in the degenerate type dynamical system. The system response of degenerate type is shown by the trajectory I in Fig.3.1, where $r=r_s$ is a singular point. In the stationary response of degenerate case, only the deterministic behavior appears, based on the drift term V(r), because the stochastic movement disappears at a singular point $r=r_s$.

In practice, the dynamical system with state-dependent noise, that is,

$$(3.5) \qquad \ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = x\dot{\xi}(t)$$

may be considered to be of degenerate type, because the differential generator of (3.4) is, though the derivation is shown afterwards, obtained by

(3.6) $$L_r(\cdot) = \frac{1}{4}r^2\sigma^2\frac{d^2(\cdot)}{dr^2} + V(r)\frac{d(\cdot)}{dr}$$

and then has a singular point $r_s=0$. Generally, the existence of $x\dot{\xi}(t)$ effects as an unstable component which obstructs the stability condition $L_rV_L \leq 0$.

The purpose of this Chapter is concerned with the condition for the existence of the stationary probability density function $p^*(r)$ and to deal with two new approaches to the analysis of nonlinear stochastic systems of degenerate type, based on the concept of an invariant measure.

In Section 3.2, as a mathematical preliminary, the relation between the transition probability density function and the invariant measure is briefly explained. The condition for the existence of an invariant measure is described in Section 3.3 by two different methods: i.e., one is Lyapunov-like function approach and another martingale approach. As examples, in Section 3.4, we show the behaviors of two kinds of nonlinear dynamical systems subjected to random excitation, including the results obtained by digital simulation studies.

3.2 Mathematical Preliminary

For convenience of discussion, let r(t) be the scalar Markov process with the differential generator,

(3.4) $$L_r(\cdot) = U^2(r)\frac{d^2(\cdot)}{dr^2} + V(r)\frac{d(\cdot)}{dr} .$$

The definition of an invariant measure is stated as follows:

[Definition 1](Dynkin[69]): For $A \in B$(Borel set), if $\mu \neq 0$ and if

(3.7) $\mu(A) = \int_{E^{(1)}} \mu(dr) P(t,r,A)$,

then the measure μ is invariant for the $r(t)$-process where the transition probability $P(t,r,A)$ is defined by

(3.8) $P(t,r,A) = P\{r(t) \in A | r(0) = r\}$.

The equality (3.7) shows the relation between the transition probability and the invariant measure.

With an additional but simple condition, it is known that there exists a probability density function $p^*(r)$ associated with $P(t,r,A)$ such that [16]

(3.9) $\mu(A) = \int_{E^{(1)}} p^*(r) dr$.

Furthermore, the probability density function $p^*(r)$ is the normalized positive solution of

(3.10) $U^2(r) \frac{\partial^2 p^*}{\partial r^2} + V(r) \frac{\partial p^*}{\partial r} = 0$,

which corresponds to the one-dimensional expression of Eq.(3.4).

From the equality (3.9), the invariant measure is strongly related to the stationary probability density function. In the sequel, our major attention is focussed on the invariant measure μ rather than the transition probability or the stationary probability density function.

3.3 Existence Conditions of an Invariant Measure

Two approaches are developed in this section, stating sufficient conditions for the existence of an invariant measure. We shall consider again the one-dimensional Markov process $r(t)$ which is called the diffusion process. In Eq.(3.4), the points r_s satisfying

(3.11) $U^2(r_s) = 0$

are called the singular points at which the diffusion motion disappears. Thus the singular points are those specified to be treated in the following discussion.

3.3.A Lyapunov-like Function Approach

We need the following two lemmas.

[Lemma 3.1] (Khas'minskii[11]) If the one-dimensional diffusion process $r(t)$ satisfies ; (a) for every initial state, any bounded region preassigned on the real line $[0,\infty)$ is hit eventually w.p.1, (b) the hitting time has a finite expectation, then there exists a unique invariant measure μ on the Borel sets of $E^{(1)}$.

Before the statement of Lemma 3.2, we introduce a real-valued function $V_L(r)$ with the following properties;

(P.1) V_L is defined for $r\in\bar{D}_V$ where $\bar{D}_V=\{r:r\geq R\}$, and where R is an arbitrary positive constant.

(P.2) V_L is continuous in $\bar{D}_V$ and is twice continuously differentiable in D_V.

(P.3) $V_L(r)\geq 0$, $r\in\bar{D}_V$ and $V_L(r)\to\infty$ as $r\to\infty$.

With the properties (P.1) to (P.3), the following lemma holds:

[Lemma 3.2] (Wonham[16]) If there exists a function $V_L(r)$ with the properties (P.1) to (P.3) and if

(3.12) $L_r V_L(r) \leq -1, \quad r\in D_V,$

then the $r(t)$-process satisfies Lemma 3.1 and has an invariant measure.

However, Lemma 3.2 can not give the condition for the existence of an invariant measure $\mu(A)$ to the nonlinear dynamical system with the differential generator (3.4), because it is still unknown whether the condition (3.11) holds for any conditions of coefficients $U^2(r)$ and $V(r)$ in (3.4) and furthermore for any functions of $V_L(r)$ with properties (P.1) to (P.3). In particular, as mentioned by Eq.(3.10), when the diffusion term $U^2(r)$ in Eq.(3.4) has the singular points, it is, in general, difficult to establish the condition for the existence of an invariant measure. Then, in this chapter, we propose an extensive approach to determine conditions of coefficients $U^2(r)$ and $V(r)$ for the existence of an invariant measure $\mu(A)$, through the construction of a Lyapunov-like function $V_L(r)$.

Suppose that, within the semi-infinite interval $0 \le r < \infty$, the differential generator (3.6) has only one singular point for which Eq.(3.11) holds.

(i) Define a new process $\zeta(t) = r(t) - r_s$ for $r_s \le r < \infty$ and write the differential generator L_ζ by

$$(3.13) \quad L_\zeta = U_\zeta^2(\zeta)\frac{d^2}{d\zeta^2} + V_\zeta(\zeta)\frac{d}{d\zeta}.$$

(ii) Similarly, define a new process $\eta(t) = r_s - r(t)$ for $0 \le r \le r_s$ and write the differential generator L_η by

$$(3.14) \quad L_\eta = U_\eta^2(\eta)\frac{d^2}{d\eta^2} + V_\eta(\eta)\frac{d}{d\eta}.$$

For the $\zeta(t)$-process defined, the following theorem holds:

[Theorem 3.1] If the coefficients of the differential generator

(3.13) satisfy the following conditions:

(C.1) $U_\zeta^2(\zeta) \neq 0$ for $\zeta > 0$.

(C.2) $\Psi(\zeta) = \Phi(\zeta)\exp\{\int_\zeta^{\zeta_0} \frac{dz}{U_\zeta^2(z)\Phi(z)}\}$ is integrable,

$$\text{where} \quad \Phi(\zeta) = \exp\{\int_\zeta^{\zeta_0} \frac{V_\zeta(z)}{U_\zeta^2(z)} dz\},$$

and ζ_0 is an arbitrary positive constant, then, the $\zeta(t)$-process has an invariant measure in the neighborhood of $\zeta=0$.

(Proof) Define the function $V_L(\zeta)$ by

$$(3.15) \quad V_L(\zeta) = \int_0^\zeta \exp[\int_y^{\zeta_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta + \int_y^{\zeta_0} \frac{1}{U_\zeta^2(z)} \exp\{-\int_z^{\zeta_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta\} dz] dy.$$

From Eq.(3.15) and the property (C.1), since

$$(3.16) \quad \frac{dV_L(\zeta)}{d\zeta} = \exp[\int_\zeta^{\zeta_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta$$

$$+ \int_\zeta^{\zeta_0} \frac{1}{U_\zeta^2(z)} \exp\{-\int_z^{\zeta_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta\} dz] > 0,$$

the function $V_L(\zeta)$ is a monotone increasing function. Hence, the function $V_L(\zeta)$ satisfies the properties (P.1) to (P.3). With the conditions (C.1) and (C.2), it follows that

$$(3.17) \quad L_\zeta V_L(\zeta) = U_\zeta^2(\zeta)\{-\frac{V_\zeta(\zeta)}{U_\zeta^2(\zeta)}\Psi(\zeta) - \frac{1}{U_\zeta^2(\zeta)}$$

$$\times \exp[\int_\zeta^{\zeta_0} \frac{1}{U_\zeta^2(z)} \exp\{-\int_z^{\zeta_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta\} dz]\} + V_\zeta(\zeta)\Psi(\zeta)$$

$$= -\exp[\int_{\zeta}^{\zeta_0} \frac{1}{U_{\zeta}^2(z)} \exp\{-\int_{z}^{\zeta_0} \frac{V_{\zeta}(\eta)}{U_{\zeta}^2(\eta)} d\eta\} dz] \leq 0.$$

For $0 \leq \zeta \leq \zeta_0$, it is apparent that

$$(3.18) \quad \int_{\zeta}^{\zeta_0} \frac{1}{U^2(z)} \exp\{-\int_{z}^{\zeta_0} \frac{V_{\zeta}(\eta)}{U_{\zeta}^2(\eta)} d\eta\} dz \geq 0.$$

Hence, it is obvious that

$$(3.19) \quad L_{\zeta} V_L(\zeta) \leq -1.$$

Consequently, by using Lemma 3.2, the $\zeta(t)$-process has an invariant measure within the interval $0 \leq \zeta \leq \zeta_0$.

For the interval $0 \leq \eta \leq r_s$, by using the differential generator (3.14), the same result as shown by the inequality (3.19) is easily obtained.

Thus the sufficient conditions for the existence of an invariant measure with respect to the $r(t)$-process are established within the semi-infinite interval $0 \leq r < \infty$.

On the other hand, for any Borel-measurable continuous function f, recognizing the fact that the equality (3.7) is equivalent to

$$(3.20) \quad \int_{E^{(1)}} f(r)\mu(dr) = \int_{E^{(1)}} T_t f(r)\mu(dr),$$

it can be stated that if there exists a measure satisfying (3.7), then it is equivalent to $P(t,r,A)$,[11] where, for all t,

$$(3.21) \quad T_t f(r) = \int_{E^{(1)}} f(x) P(t,r,dx),$$

where T_t is a linear operator which forms a contraction semi-group such that, for each fixed r, the function $T_t f(r)$ is continuous in t, and then has the following properties; (1) positive preserving, (2) $\| T_t \| \le 1$. This fact stimulates us to originate another approach termed the martingale approach.

3.3.B Martingale Approach

This approach is based on the following lemma. First of all, our attention is directed to the $\zeta(t)$-process with the differential generator L_ζ defined by Eq.(3.13).

[Lemma 3.3] (Beneš[12]) If

$$(3.22) \quad \int_{E^{(1)}} T_t f(\zeta)\mu(d\zeta) = \int_{E^{(1)}} f(\zeta) T_t^* \mu(d\zeta) < \infty ,$$

the $\zeta(t)$-process has an invariant measure, where T_t^* is the adjoint operator of T_t.

Furthermore, we need the following lemma.

[Lemma 3.4] (Doob[70]) Let the $\zeta(t)$-process be an Itô-type stochastic process with bounded diffusion coefficients, $0<U_\zeta^2(\zeta)<\infty$. Suppose that

$$(3.23) \quad E\{\exp|\zeta(0)|\} < \infty.$$

If

$$(3.24) \quad L_\zeta f = 0,$$

then the corresponding stochastic process $\{f[\zeta(t)], t\ge 0\}$ is a martingale.

However, it is, in general, difficult to construct the martingale function $f[\zeta(t)]$ satisfying the condition (3.24) in Lemma 3.4. The method proposed here is to find sufficient conditions for the existence of an invariant measure $\mu(A)$, through the construction of a martingale function $f[\zeta(t)]$. The following two theorems are stated, based on lemmas mentioned above.

[Theorem 3.2] Let $\zeta(t)$ be a stochastic process of Itô-type and $E\{\zeta(0)\}<\infty$. If the function $f(\zeta)$ defined by

$$(3.25)\quad f[\zeta(t)] = \int_0^{\zeta} \exp\{\int_z^{r_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta\} dz$$

is bounded and continuous for $0\le\zeta<\infty$, then $f[\zeta(t)]$ is a martingale.

(Proof) The $\zeta(t)$-process is a stochastic process of Itô-type with the differential generator (3.13). From the definition (3.20), it is apparent that $f[\zeta(t)]$ is a continuous and twice continuously differentiable in $0\le\zeta<\infty$. Furthermore, using (3.13), it follows that

$$(3.26)\quad L_\zeta f = U_\zeta^2(\zeta)[-\frac{V_\zeta(\zeta)}{U_\zeta^2(\zeta)} \exp\{\int_\zeta^{r_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta\}]$$

$$+ V_\zeta(\zeta)\exp\{\int_\zeta^{r_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta\} = 0.$$

Then, with the help of Lemma 3.4, it is obvious that $f[\zeta(t)]$ is a martingale.

[Theorem 3.3] If Theorem 3.2 holds and

$$(3.27)\quad f[\zeta(0)] = \int_0^{\zeta(0)} \exp\{\int_z^{r_0} \frac{V_\zeta(\eta)}{U_\zeta^2(\eta)} d\eta\} dz < \infty,$$

then, there exists an invariant measure $\mu(A)$ for the $\zeta(t)$-process.

(Proof) Set

(3.28) $T_t^* \mu(A) = P(t,r,A)$.

Then, from (3.22), we have

(3.29) $\int_{E^{(1)}} f(\zeta) T_t^*(d\zeta) = \int_{E^{(1)}} f(\zeta) P(t,\zeta,d\zeta)$.

If the function $f(\zeta)$ is given by Eq.(3.25), it is obvious that

(3.30) $E\{f[\zeta(t)] \,|\, \zeta(0)=\zeta\} = f[\zeta] < \infty$,

because $f[\zeta(t)]$ is a martingale. This implies that the inequality (3.22) holds. Hence, the $\zeta(t)$-process has an invariant measure $\mu(A)$.

For the $\eta(t)$-process with the differential generator (3.14), an analogous theorem to Theorem 3.3 is easily stated.

3.4 Illustrative Examples

We shall consider the second-order nonlinear stochastic differential equation,

(3.31) $\ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = \delta h(x,\dot{x})\dot{\xi}(t)$

Equation (3.31) may be considered to be a generalization of a mathematical model of dynamical systems, where $g(x,\dot{x})$ expresses the system nonlinearity, $\dot{\xi}(t)$ is a Gaussian white noise process, and $h(x,\dot{x})$ is a nonlinear function by which various kinds of excitation $h(x,\dot{x})\dot{\xi}(t)$ are generated. The nonlinearities $g(x,\dot{x})$ and $h(x,\dot{x})$ contain both velocity and displacement terms and may depend on the past history of the system. It is assumed that ω^2, ε and δ are constants and that both ε and δ are small in some sense such that the system is lightly damped, weakly nonlinear and that the system response is related to a random excitation with a relatively small magnitude.

Let the state variables be $x=x_1$ and $\dot{x}_1=x_2$ respectively. Then, Eq.(3.31) is expressed by the nonlinear stochastic differential equation of Itô-type[66],[71],

$$\text{(3.32a)}\quad dx_1 = x_2 dt$$

$$\text{(3.32b)}\quad dx_2 = -\{\omega^2 x_1 + \varepsilon g(x_1,x_2)\}dt + \delta h(x_1,x_2)dw(t)$$

where the w(t)-process is a Brownian motion process and this has been introduced through the relation[72],

$$\text{(3.33)}\quad w(t) = \int_0^t \dot{\xi}(s)ds.$$

Naturally, the following properties are well-known ; $E\{dw(t)\}=0$, $E\{(dw(t))^2\}=\sigma^2 dt$.

In order to convert the two-dimensional stochastic process (x_1,x_2) determined by Eqs.(3.32a) and (3.32b) into the one-dimensional stochastic process r(t), letting

$$\text{(3.34)}\quad x_1 = \frac{r}{\omega}\sin(\psi-\omega t),\quad x_2 = -r\cos(\psi-\omega t),$$

then,

$$\text{(3.35)}\quad r^2(t) = \omega^2 x_1^2(t) + x_2^2(t)$$

and after somewhat tedious calculations using the averaging principle[46] (for more detail, see Ref.[51]), the following differential generator can be obtained;

$$\text{(3.36)}\quad L_r = U^2(r)\frac{d^2}{dr^2} + V(r)\frac{d}{dr},$$

where

$$(3.37)\quad U^2(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\sin^2\theta d\theta,$$

$$(3.38)\quad V(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\frac{\cos^2\theta}{r}d\theta$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} g(\frac{r}{\omega}\cos\theta,\ r\sin\theta)\sin\theta d\theta$$

and $\psi-\omega t = \theta+\pi/2$, and where it has been assumed that $\delta^2=\varepsilon$.

3.4.A Example-1

Let the nonlinear functions $g(x,\dot{x})$ and $h(x,\dot{x})$ be given by

$$(3.39)\quad g(x,\dot{x}) = g(x_1,x_2) = x_1^3 + 2\alpha x_2$$

and

$$(3.40)\quad h(x,\dot{x}) = h(x_1,x_2) = \gamma$$

respectively, where α and γ are constants. With Eqs.(3.39) and (3.40), we have

$$(3.41a)\quad dx_1 = x_2 dt$$

$$(3.41b)\quad dx_2 = -\{\omega^2 x_1 + \varepsilon(x_1^3 + 2\alpha x_2)\}dt + \delta\gamma dw.$$

Equation (3.41) is a mathematical model of dynamical systems with the nonlinear restoring force of cubic order and excited by a Gaussian white noise. We set γ as a positive constant. When $\dot{\xi}(t)=0$, i.e., $dw=0$, it is already known that the system is stable, provided that $\alpha>0$, while the system is unstable, if $\alpha<0$.

From (3.37), it can easily be shown that

$$(3.42)\quad U^2(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} \gamma^2\sin^2\theta d\theta = \frac{\sigma^2\gamma^2}{4}$$

Hence, no singular points exist. Furthermore, the fact that the system (3.41) has no invariant measures is a direct consequence of the application of Theorem 3.1 and 3.3. In fact, since

$$(3.43)\quad V(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} \gamma^2 \frac{\cos^2\theta}{r} d\theta - \alpha r$$

$$= \frac{\sigma^2\gamma^2}{4r} - \alpha r,$$

it follows that

$$(3.44)\quad \int_0^{\zeta} \Phi(r)dr = \int_0^{\zeta} A_0 r^{-1}\exp(\frac{2\alpha}{\sigma^2\gamma^2}r^2)dr = \infty$$

and

$$(3.45)\quad \int_0^{\zeta} \Psi(r)dr = \int_0^{\zeta} A_0 r^{-1}\exp(\frac{2\alpha}{\sigma^2\gamma^2}r^2)$$

$$\times \exp\{\int_r^{r_0} \frac{x}{\sigma^2\gamma^2}\exp(\frac{-2\alpha}{\sigma^2\gamma^2}x^2)dx\}dr = \infty$$

where $A_0 = \zeta_0$.

3.4.B <u>Example-2</u>

Instead of (3.40), letting

$$(3.46)\quad h(x,\dot{x}) = h(x_1,x_2) = \gamma x_1,$$

then, it may happen that there exists a singular point. Thus, we can clearly understand the behavior of sample trajectories of the system, based on the existence of an invariant measure and the related stationary probability density function. The system equation is

$$(3.47a)\quad dx_1 = x_2 dt$$

$$(3.47b)\quad dx_2 = -\{\omega^2 x_1 + \varepsilon(2\alpha x_2 + x_1^3)\}dt + \delta\gamma x_1 dw.$$

Equation (3.47) is a mathematical model of dynamical and/or control systems excited by a random noise which is proportional to displacement of the system. Another practical example is considered to be a mathematical model of nonlinear systems containing a random coefficient because, from (3.47), we may write

$$(3.48)\quad \ddot{x} + 2\varepsilon\alpha\dot{x} + \{\omega^2 - \delta\gamma\dot{\xi}(t)\}x + \varepsilon x^3 = 0.$$

From (3.47) and (3.48), the diffusion and the drift coefficients are respectively computed to be

$$(3.49)\quad U^2(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} \frac{\gamma^2}{\omega^2} r\cos^2\theta\sin^2\theta d\theta$$

$$= \frac{\sigma^2\gamma^2}{16\omega^2} r^2$$

$$(3.50)\quad V(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} \frac{\gamma^2}{\omega^2} r^2\cos^4\theta\frac{1}{r}d\theta - \alpha r$$

$$= (\frac{\sigma^2\gamma^2}{16\omega^2} - \alpha)r.$$

From Eq.(3.49), the singular point is $r_s = 0$, i.e., the origin. In particular, this point is a trap because $V(0)=0$. With the help of (3.49) and (3.50), $\Phi(r)$ and $\Psi(r)$ are respectively

$$(3.51)\quad \Phi(r) = \exp\{\int_r^{r_0} [(\frac{\sigma^2\gamma^2}{16\omega^2} - \alpha)\Big/\frac{\sigma^2\gamma^2}{16\omega^2}x]dx\}$$

and

$$(3.52)\quad \Psi(r) = A_0 r^{-1+16\alpha\omega^2/\sigma^2\gamma^2}\exp\{\frac{16\omega^2}{\sigma^2\gamma^2}\int_r^{r_0} x^{-1+16\alpha\omega^2/\sigma^2\gamma^2}dx\}$$

where $A_0=r_0^{1-16\alpha\omega^2/\sigma^2\gamma^2}$.

(A) The case where $(16\alpha\omega^2/\sigma^2\gamma^2) \leq 0$ $(\alpha \leq 0)$

With the help of (3.51) and (3.52), examinations of (C.2) and (3.27) become respectively,

$$(3.53) \quad \int_0^\zeta \Phi(r)dr = A_0 \int_0^\zeta r^{-1+16\alpha\omega^2/\sigma^2\gamma^2} dr = \infty$$

and

$$(3.54) \quad \int_0^\zeta \Psi(r)dr = A_0 \int_0^\zeta r^{-1+16\alpha\omega^2/\sigma^2\gamma^2} \exp\{\frac{16\omega^2}{\sigma^2\gamma^2} \times \int_r^{r_0} x^{-1-16\alpha\omega^2/\sigma^2\gamma^2} dx\}dr = \infty \ .$$

Then, Theorems 3.1 and 3.3 do not hold. Consequently, in this case, since the stationary probability density function does not exist and Eq.(3.10) may not be applicable.

(B) The case where $(16\alpha\omega^2/\sigma^2\gamma^2) > 0$ $(\alpha > 0)$

In this case, we have

$$(3.55) \quad \int_0^\zeta \Psi(r)dr = \infty$$

and

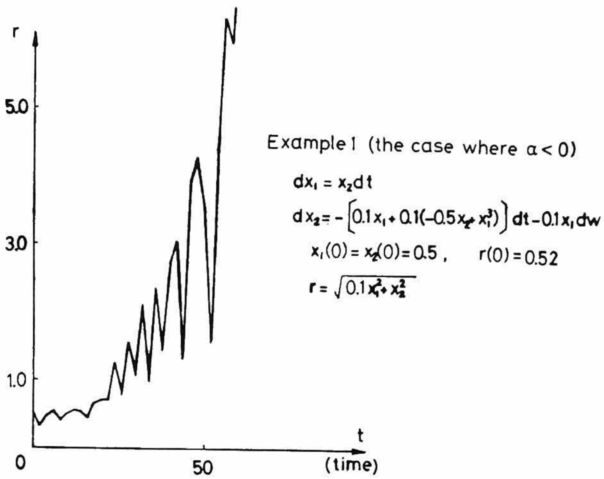
$$(3.56) \quad f[\zeta(0)] < \infty \ .$$

Thus, although Theorem 3.1 does not hold, Theorem 3.3 holds. This implies that Theorem 3.1 gives a sufficient condition and strongly depends on a choice of Lyapunov-like functions. However, bearing in mind Lemma 3.3, it can be concluded that if $16\alpha\omega^2/\sigma^2\gamma^2 > 0$, then the system has an invariant measure and sample trajectories approach the origin. This is also examined by digital simulation studies in the sequel. Thus, the existence of the stationary probability density function $p^*(r)$ depends on whether the value of $\kappa=16\alpha\omega^2/\sigma^2\gamma^2$ is positive or not. However, it should be noted that, from (3.51), we have

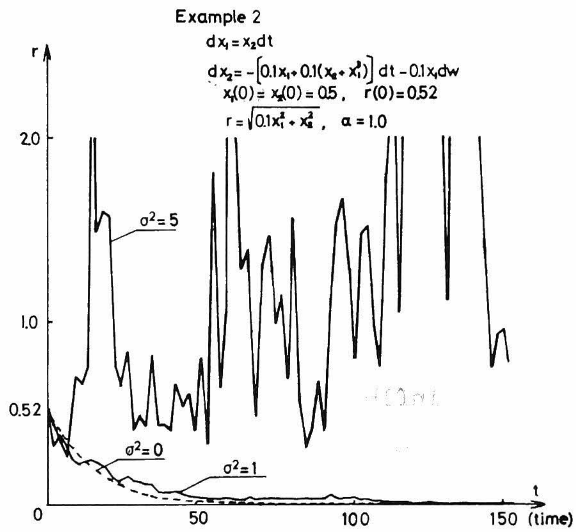
$$(3.57) \quad \lim_{\sigma^2\to\infty} \int_0^\zeta \Phi(r)dr = \lim_{\sigma^2\to\infty} A_0 \int_0^\zeta r^{-1+16\alpha\omega^2/\sigma^2\gamma^2} dr$$

$$= \infty \ .$$

It is obvious that , as the value of σ^2 or γ^2 becomes larger, the value of κ becomes smaller. The digital simulation studies are performed on Eq.(3.47). Naturally, the value of κ does not indicate a critical value but gives an estimate guaranteeing the existence of an invariant measure, i.e., the stationary solution $p^*(r)$ of the Fokker-Planck equation. It is worthwhile to show the sample trajectories obtained by digital simulation studies. A wide variety of sample runs determined by Eq.(3.47) were simulated on a digital computer. Figure 3.2 shows a representation of sample runs where, through the experiments, a constant step-size of time intervals was taken as $\Delta t=0.01$(sec). A set of parameter values was preassigned as $\omega^2=0.1$, $\varepsilon=0.1$ and $\delta=0.3$. Figure 3.2(a) shows a sample run in the case where $\alpha<0$ for which the existence of the stationary probability density function $p^*(r)$ can not be expected and the system without an exciting force is inherently unstable. Three sample runs are plotted in Fig.3.2(b). The dotted line, $\gamma=0$, corresponding to the deterministic trajectory with initial value $r_0=0.52$. The initial state in the case in which the system was excited by a random input with a variance $\sigma^2=1$ was also $r_0=0.52$. Through the simulation experiments, in the case in which the system was excited by a random force with a small magnitude, the sample



(a) Damping coefficient $\alpha = -0.5$



(b) Damping coefficient $\alpha = 1.0$

Fig.3.2 Sample path behavior of the system given by Eq.(3.47).

runs converge to the singular point $r_s=0$. On the contrary, as the magnitude of a random exciting force becomes larger, sample runs do not bring any information to conclude the existence of a stationary probability density function. This can be seen in Eq.(3.57), stating the fact that as the magnitude of an exciting force becomes larger, the possibility of the existence of the stationary probability density functions becomes remote.

3.4.C Application to the Noise Stabilization Problem

Two examples presented above have a singular point at the origin r=0. However, there exist many other cases where the system has a singular point at $r=r_s \neq 0$. Practical examples have already been shown by the authors associated with studies on the noise stabilization of nonlinear systems[51],[52]. For example, we shall consider again the nonlinear dynamical system given by

$$(3.58) \quad \ddot{x} + \omega^2 x + \varepsilon(\alpha x + \beta x^3) = -\delta h(x,\dot{x})\dot{\xi}(t)$$

where

$$(3.59) \quad h(x,\dot{x}) = h(r) = ar|r - r_s|.$$

When $\dot{\xi}(t)=0$, the system behavior shows the limit cycle which is regarded as the unstable state. Our purpose is to eliminate the limit cycle by using the influence of the additional term $h(x,\dot{x})$ $\times\dot{\xi}(t)$. In this study, a key assumption is the existence of the stationary probability density function p^*. It becomes possible to justify this assumption by applying the approach presented here. The system has a singular point $r=r_s$ where r_s is the preassigned location of a singular point for the purpose of realizing the stabilization. Then, we shall examine the behavior of the processes in the two intervals $I_\eta = [0,r_s]$ and $I_\zeta = [r_s,\infty)$. First, in the interval I_ζ, Eq.(3.13) becomes

(3.60) $$L_\zeta = \frac{\sigma^2 a^2}{4}(\zeta + r_s)\zeta\frac{d^2}{d\zeta^2} + \frac{\sigma^2 a^2}{4}\zeta\frac{d}{d\zeta} .$$

According to the same procedure as cases of 3.4.A and 3.4.B, we have

(3.61) $$\int_0^\zeta \Phi(\eta)d\eta = \int_0^\zeta \frac{K_1}{\eta+r_s}d\eta < \infty,$$

and

(3.62) $$\int_0^\zeta \Psi(\eta)d\eta = \int_0^\zeta \frac{K_1K_2}{\eta^\alpha(\eta+r_s)}d\eta < \infty$$

where $K_1=\zeta_0+r_s$, $\alpha= 4/\sigma^2a^2K_1$ and $0\leq\alpha\leq 1$ and where $K_2=\zeta_0^{(4/\sigma^2a^2(\zeta_0+r_s))}$.

From the results of Eqs.(3.61) and (3.62), it is obvious that Theorems 3.1 and 3.3 hold. Furthermore, it is a simple exercise to obtain the same result in the I_η interval as in the interval I_ζ. Consequently, it can be concluded that the behavior of the r(t)-process converges to the singular point $r=r_s$ w.p.1. Simulation results were already shown in [51].

3.5 Summary

In this chapter, two new approaches have been developed to examine the existence of the stationary probability density function for the response of nonlinear dynamical systems. One is to choose a suitable Lyapunov-like function and another to find out an arbitrary martingale function. Both approaches presented here give sufficient conditions of guaranteeing the existence of the stationary solution of the Fokker-Planck equation. Naturally, a choice of a type of Lyapunov-like functions depends on the nonlinear characteristics contained in the dynamical system considered. However, the present methods provide an exploration of the asymptotic behavior of a wide class of nonlinear dynamical systems around the singular points. There is an additional advantage to the present methods, which can be used to find out the possibility of realizing the noise stabilization of unstable nonlinear dynamical systems, which will be described in Chapter 6.

CHAPTER 4

STOCHASTIC BEHAVIORS OF NONLINEAR DYNAMICAL SYSTEMS OF NON-DEGENERATE TYPE

4.1 Introduction

We shall again consider the r(t)-process with the following differential generator,

$$(4.1) \qquad L_r(\cdot) = U^2(r)\frac{d^2(\cdot)}{dr^2} + V(r)\frac{d(\cdot)}{dr},$$

which was obtained by the nonlinear stochastic differential equation (3.31) or (3.32). (for more detail, see Section 3.4) In Chapter 3 already presented, for the nonlinear dynamical system of degenerate type with $U^2(r_s)=0$ in (4.1), sufficient conditions were shown that the r(t)-process converges to a singular point r_s in the stationary states.

In this Chapter, we shall consider stochastic behaviors of nonlinear dynamical systems of non-degenerate type with $U^2(r)\neq 0$ for all r in (4.1). The system behavior of non-degenerate type is shown by the trajectory I in Fig.4.1, where a point $r=r_e$ satisfies

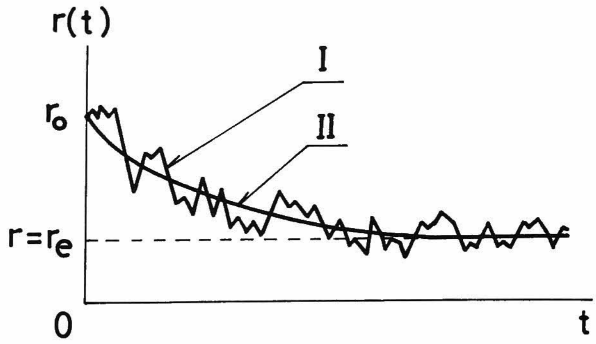


Fig.4.1 System Behavior of Non-degenerate Type

$V(r_e)=0$ but $U^2(r_e)\neq 0$ and the trajectory II represents the deterministic behavior converging to a point $r=r_e$. Accordingly, since there is always $U^2(r)\neq 0$ for all r, the system behavior of non-degenerate case is that the stochastic movement is added to the deterministic dynamics II , whether the system is stable (in this case, the r(t)-process converges stochastically at $r=r_e$ as $t\to\infty$) or unstable.

In practice, as the structure of dynamical systems with an external noise, there exist a lot of systems of non-degenerate type. For example, the dynamical system with an additive noise as $h(x,\dot{x})=c$(=constant) in Eq.(3.31), that is,

$$(4.2) \qquad \ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = c\dot{\xi}(t)$$

may be considered to be of non-degenerate type, because the differential generator of (4.1) is, using Eqs.(3.37) and (3.38), obtained by

$$(4.3) \quad L_r(\cdot) = \frac{1}{4}c^2\sigma^2\frac{d^2(\cdot)}{dr^2} + V(r)\frac{d(\cdot)}{dr}$$

and then (4.3) has no singular points.

Comparing with the case of a state-dependent noise $x\dot{\xi}(t)$, the additive noise $c\dot{\xi}(t)$ operates as an undeterministic component to the system dynamics. Accordingly, in the non-degenerate case, it is an important problem how these dynamical systems, corresponding to the level of random noise, converge stochastically to some domain under any conditions in the stationary states.

In Section 4.2, the description of the problem to be solved here is explained. In Section 4.3, for behaviors of nonlinear dynamical systems of non-degenerate type, two main theorems are demonstrated giving sufficient conditions for the existence of the stationary response and for the convergence of sample trajectories to the stationary state with a certain probability appraisal. As illustrative examples, in Section 4.4, we show the behavior of two kinds of nonlinear dynamical systems subjected to random excitation, including the results obtained by digital simulation studies.

4.2 Problem Statement

As a mathematical preliminary, the relation between the transition probability density function and the invariant measure was already explained in Chapter 3. The purpose of this chapter is to examine both the existence of the stationary response and the asymptotic behavior of nonlinear stochastic systems that the $r(t)$-process which starts from the given initial states converges to some domain with a certain probability when $t\to\infty$. Figure 4.1 illustrates the analytical procedure which will be developed in the

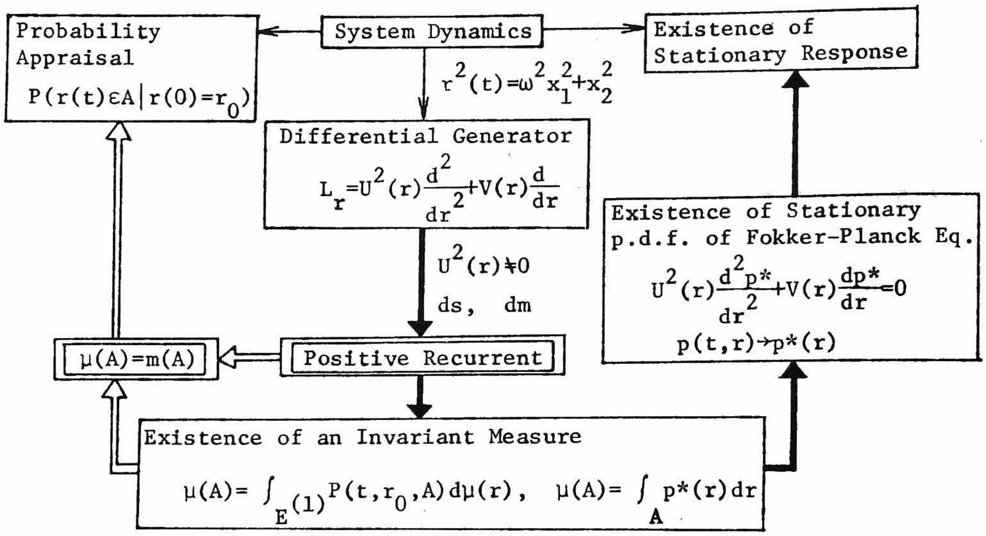


Fig.4.2 Illustration of Analytical Procedure

sequel. First, as shown in Fig.4.2, the differential generator L_r of the r(t)-process will be found by applying the averaging principle by Khas'minskii to the Kolmogorov equation derived by using the polar coordinate transformation. Secondly, the concept of positive recurrent related to the knowledge of sample properties is introduced, by which the sufficient conditions for the existence of an invariant measure μ(A) will be obtained. This result implies that there exists the stationary probability density function $p^*(r)$ with respect to the r(t)-process. Finally, applying the relation that the canonical measure m(r) is equivalent to an invariant measure μ(A), the probability appraisal P(t,r,A) that sample behaviors converge to some domain will be examined.

4.3 Existence Conditions of an Invariant Measure and Analysis of System Behaviors

We shall consider the one-dimensional Markov process $r(t)$ of non-degenerate type. In Eq.(4.1), the sets of all non-singular points satisfying

$$(4.4) \qquad U^2(r) \neq 0$$

are particularly called the non-singular intervals which are either semifinite or finite. Now we take any non-singular interval $I' = [r_i, r_j]$ on which Feller's scale ds and the speed measure dm are given as [65],[73]

$$(4.5) \qquad dm(r) = \frac{1}{U^2(r)} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)} d\eta\} dr$$

and

$$(4.6) \qquad ds(r) = \exp\{-\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)} d\eta\} dr,$$

where r_a is an arbitrary point assigned in I', $s(r)$ a right-continuously increasing function and $m(r)$ continuously increasing function.

Using Eqs.(4.5) and (4.6), the following two lemmas hold, which are already known.

[Lemma 4.1] (K.Itô and M.Nisio[14]) If the canonical scale ds and the speed measure dm are characterized by

$$(4.7) \qquad \int_0^{\infty} ds(r) = \infty, \quad \int_{r_0}^{\infty} ds(r) = \infty \quad \text{and} \quad \int_0^{\infty} dm(r) < \infty$$

where r_0 is an arbitrary point in $I=[0,\infty)$, then the $r(t)$-process is positive recurrent.

[Lemma 4.2] If there exists a non-singular finite interval $I' = [r_i, r_j]$ such that, for the drift term $V(r)$ of the differential generator (4.1), $V(r_i)>0$ and $V(r_j)<0$, then the r(t)-process is positive recurrent.

Since the proof is straightforward by [14], the description of the proof is omitted here. From the lemmas 4.1 and 4.2, the following fact is already known. [14],[74]

[Lemma 4.3] If the r(t)-process is positive recurrent in the non-singular interval I or I', there exists an invariant measure $\mu(A)$.

Based on the lemmas 4.1 and 4.3, the following theorem holds.

[Theorem 4.1] Let the r(t)-process be with the differential generator (4.1). For Eq.(4.1), it is assumed that

(C.1) $U^2(r)=a(=\text{constant})>0$ in $r\in[0,\infty)$

(C.2) $V(r)$ is a continuous and bounded function for all r except for r=0.

(C.3) The function $\exp\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\}$ is continuous and bounded in $r\in[0,\infty)$, has $O(1/r^n)$ as $t\to\infty$ and, furthermore,

$$\lim_{r\to\infty} \exp\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\} = 0.$$

Then, there exists an invariant measure $\mu(A)$.

(Proof) First, the speed measure dm is examined. The integral in Eq.(4.7) may be

$$(4.8) \qquad \int_0^\infty dm(r) = \int_0^{r_L} dm(r) + \int_{r_L}^{\infty} dm(r)$$

where $r=r_L$ is sufficiently large and $r_a<r_L$. From the conditions (C.1),(C.2) and (C.3), it is obvious that dm(r) is continuous and bounded in $[0,r_L]$ and then the following result is obtained;

$$(4.9) \quad \int_0^{r_L} dm(r) = \int_0^{r_L} \frac{1}{a}\exp\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\}dr < \infty.$$

Nextly, we shall consider the second term of the R.H.S. of the integral (4.8). As any function $\psi(r)$ which is positive in the interval $r_L<r$ and integrable at $r=\infty$, we set

$$(4.10) \quad \psi(r) = 1/r^n \quad (n>1).$$

Then, it is obvious that

$$(4.11) \quad \int_{r_L}^{\infty} \psi(r)dr = \int_{r_L}^{\infty} \frac{1}{r^n}dr = \frac{1}{(n-1)r_L^{n-1}} = K$$

where K is positive constant. If the following relation,

$$(4.12) \quad \frac{1}{a}\exp\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\} \leq \frac{1}{r^n} \quad (r_a \leq r)$$

holds for a sufficiently large r, it becomes that

$$(4.13) \quad \int_{r_L}^{\infty} \frac{1}{a}\exp\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\}dr \leq \int_{r_L}^{\infty} \frac{1}{r^n}dr$$
$$= K < \infty.$$

Accordingly, we shall examine whether Eq.(4.12) holds or not. Now, letting

$$(4.14) \quad F(r) = \frac{1}{r^n} - \frac{1}{a}\exp\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\}$$

and using the condition (C.3), it can be obtained that

$$(4.15)\quad \lim_{r\to\infty} F(r) = \lim_{r\to\infty} \left[\frac{1}{r^n} - \frac{1}{a}\exp\left\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\right\}\right]$$

$$= 0$$

and

$$(4.16)\quad dF(r)/dr = 0.$$

From Eqs.(4.15) and (4.16), $F(r)>0$. Then, Eq.(4.12) holds. From the results of Eqs.(4.9) and (4.13), it follows that

$$(4.17)\quad \int_0^\infty dm(r) = \int_0^\infty \frac{1}{a}\exp\left\{\frac{1}{a}\int_{r_a}^{r} V(\eta)d\eta\right\}dr < \infty.$$

Nextly, Feller's canonical scale ds is examined similarly as in the case of dm. Applying (C.3) to the calculation of ds, we obtain

$$(4.18)\quad \int_0^\infty ds(r) = \int_0^\infty \exp\left\{-\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)}d\eta\right\}dr = \infty.$$

Also, it is obvious that

$$(4.19)\quad \int_{r_a}^\infty ds(r) = \infty.$$

From the results (4.17), (4.18) and (4.19), since Lemma 4.1 holds and the r(t)-process is positive recurrent, we may conclude from Lemma 4.3 that there exists an invariant measure $\mu(A)$.

Theorem 4.1 reveals that there exists an invariant measure $\mu(A)$ within the interval $I=[0,\infty)$. Consequently, we may conclude that the stationary sample trajectories of the r(t)-process exist within the interval I. As shown in Fig.4.3, we shall consider behaviors of the r(t)-processes in detail. In the case of a deterministic system, the sample trajectory converges to the point $r=r_c$ as shown in Fig.4.3(a), provided that the system is stable and the

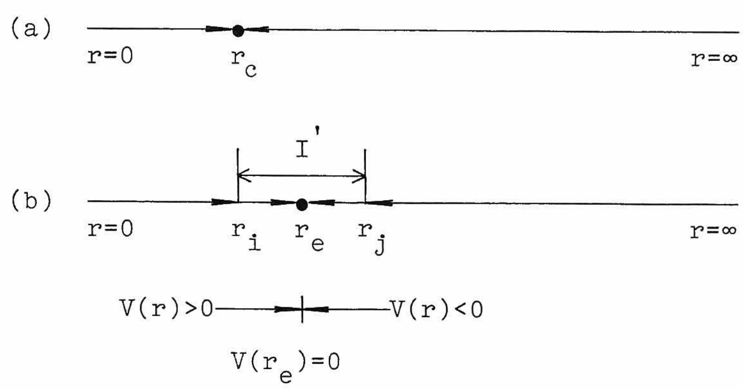


Fig.4.3 Concept of Positive Recurrent

point r_c is the equilibrium point. On the other hand, in the case of stochastic systems, this equilibrium point $r=r_c$ represents a stable singular point corresponding to both $U^2(r_c)=0$ and $V(r_c)=0$. However, in the case of nonlinear stochastic systems of non-degenerate type considered here, there exists only a point r_e which satisfies $U^2(r)\neq 0$ but $V(r)=0$. Let this point be $r=r_e$, as shown in Fig.4. (b). Naturally, if no random excitation exists, then $r_e=r_c$. We shall consider a finite non-singular interval $I'=[r_i,r_j]$ including the point $r=r_e$. If there exists a point $r=r_e$ satisfying $V(r_e)=0$ and if, in the neighborhood of $r=r_e$, $V(r_i)>0$ at $r=r_i$ and $V(r_j)<0$ at $r=r_j$, it follows from Lemma 4.2 that sample trajectories converges to to $r=r_e$ in I' as shown in Fig.4.3 (b), wherever they start with any initial values. In this case, by giving the probability appraisal which sample trajectories of the $r(t)$-process converge to the interval I', nonlinear system behaviors of the non-degenerate type can be examined clearly within the non-singular interval I. In the present chapter, through the relation between the canonical measure

$m(r)$ defined by (4.5) and an invariant measure $\mu(A)$ given by (3.7), an approach has been developed to obtain the transition probability $P(t,r_0,I')$ which the $r(t)$-process converges to the interval I'.

We need the following two lemmas. [11],[75],[76]

[Lemma 4.4] If $\mu(E^{(1)})<\infty$, then for any $A\epsilon B$, it holds that

$$(4.20)\qquad P(t,r_0,A) = \frac{\mu(A)}{\mu(E^{(1)})}, \qquad r_0 \in E^{(1)}$$

where $E^{(1)}=[0,\infty)$.

[Lemma 4.5] The speed measure $m(A)$ is equivalent to an invariant measure $\mu(A)$, that is,

$$(4.21)\qquad \mu(A) = \int_{E^{(1)}} P(t,r_0,A)dm(r) = m(A).$$

Based on the lemmas 4.2, 4.4 and 4.5, the following theorem holds :

[Theorem 4.2] Take any finite non-singular interval $I'=[r_i,r_j]\epsilon I$. If Theorem 4.1 holds and if, for $V(r)$ of Eq.(4.1), the following assumptions hold :

(C.4) there exists only one point $r=r_e$ such that $V(r_e)=0$ in the interval I'.

(C.5) for any point $r<r_e$ in I', $V(r) > 0$ and for $r>r_e$, $V(r)<0$,

then, the $r(t)$-process converges to the point $r=r_e$ within I'. The probability that the $r(t)$-process converges within I', is given as

$$(4.22)\qquad P_r(r(t)\epsilon I' | r(0)=r_0)$$

$$= \int_{r_i}^{r_j} \frac{1}{U^2(r)} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)} d\eta\} dr \Big/ \int_0^{\infty} \frac{1}{U^2(r)} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)} d\eta\} dr$$

(Proof) By Theorem 4.1, in the non-singular interval $I=[0,\infty)$, the $r(t)$-process has an invariant measure $\mu(A)$. From (C.4) and (C.5), the $r(t)$-process satisfies Lemma 4.2. Then, the $r(t)$-process is positive recurrent in the non-singular finite interval $I'=[r_i,r_j]$ and then converges to I'. Using Lemma 4.4, the probability $P(t,r_0,I')$ which converges to I' is given by

$$(4.23) \quad P_r(r(t)\epsilon I' \mid r(0)=r_0) = \mu(I')/\mu(I).$$

Furthermore, using Lemma 4.5 and Eq.(4.5), it follows that

$$(4.24) \quad P_r(r(t)\epsilon I' \mid r(0)=r_0) = m(I')/m(I)$$

$$= \int_{I'} dm(r) \Big/ \int_I dm(r)$$

$$= \int_{r_i}^{r_j} \frac{1}{U^2(r)} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)} d\eta\} dr \Big/ \int_0^{\infty} \frac{1}{U^2(r)} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)} d\eta\} dr.$$

4.4 Illustrative Examples

4.4.A Example-1

Let the nonlinear function $g(x,\dot{x})$ and $h(x,\dot{x})$ be given by

$$(4.25) \quad g(x,\dot{x}) \equiv g(x_1,x_2) = x_1^3 + 2\alpha x_2$$

and

$$(4.26) \quad h(x,\dot{x}) \equiv h(x_1,x_2) = 1$$

respectively, where α is a constant. With (4.25) and (4.26), the system equation is

(4.27a) $dx_1 = x_2 dt$

(4.27b) $dx_2 = -\{\omega_1^2 x_1 + \varepsilon(x_1^3 + 2\alpha x_2)\}dt + \delta dw.$

Equation (4.27) is a mathematical model of dynamical systems with the nonlinear restoring force of cubic order and excited by white Gaussian noise. When $dw(t)=0$, it is already known [77] that the system is stable, provided that $\alpha>0$, while the system is unstable, if $\alpha<0$.

From Eq.(4.27), the $r(t)$-process is the scalar Markov process with the differential generator,

$$(4.28) \qquad L_r(\cdot) = U^2(r)\frac{d^2(\cdot)}{dr^2} + V(r)\frac{d(\cdot)}{dr}$$

where, using Eqs.(4.2) and (4.3), the diffusion and drift coefficients are respectively computed as

$$(4.29) \qquad U^2(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta d\theta = \frac{\sigma^2}{4}$$

and

$$(4.30) \qquad V(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^2\theta}{r}d\theta = \frac{\sigma^2}{4r} - \alpha r.$$

From Eq.(4.29), since $a=\sigma^2/4$, there exists no singular point because $U^2(r)\neq 0$. With the help of Eqs.(4.29) and (4.30), we shall examine the conditions of Theorem 4.1 ;

(i) (4.29) and (4.30) satisfy the conditions (C.1) and (C.2) respectively.

(ii) Examinations of (C.3) become,

$$(4.31)\quad \lim_{r\to 0} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)}d\eta\} = \lim_{r\to 0} R_0 r \exp(-\frac{2\alpha}{\sigma^2}r^2)$$

$$= 0 \ (\text{for } \alpha \gtrless 0)$$

and furthermore

$$(4.32)\quad \lim_{r\to\infty} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)}d\eta\} = 0 \ (\text{for } \alpha>0)$$

$$= \infty \ (\text{for } \alpha<0)$$

where $R_0=\exp(2\alpha r_a^2/\sigma^2)/r_a$. These results imply that the condition (C.3) is satisfied only when $\alpha>0$. Consequently, the system of Eq.(4.27) has an invariant measure $\mu(A)$ if and only if $\alpha>0$ and then Eq.(4.27) has the stationary response.

In order to clarify the characteristics of the stationary response, we shall apply Theorem 4.2 to Eq.(4.27). From Eq.(4.30), $V(r_e)=0$ holds only when

$$(4.34)\quad r = r_e = \sqrt{\sigma^2/4\alpha}\ .$$

Considering the non-singular interval $I'=[r_i,r_j]$ with $r=r_e$ in (4.34) where $r_i\le r_e$ and $r_j\ge r_e$, this interval I' satisfies the condition (C.5) of Theorem 4.2. Furthermore, with the help of (4.29) and (4.30), the calculation of Eq.(4.22) becomes,

$$(4.35)\quad P_r(r(t)\in I'\,|\,r(0)=r_0) = \exp(-\frac{2\alpha}{\sigma^2}r_i^2)\{1-\exp[-\frac{2\alpha}{\sigma^2}(r_j^2-r_i^2)]\}.$$

If the values r_i and r_j are fixed, then the probability $P_r(\cdot)$ can be evaluated which the r(t)-process converges to the non-singular interval I' in the stationary states.

The validity of the theoretical results obtained above is shown through digital simulation studies. Figures 4.4 and 4.5 show a representative of sample runs determined by Eq.(4.27), in the case where $r_e = 0.16$ in Fig.4.4, while a sample run in Fig.4.5 is in the case where $r_e=0.25$, respectively. Through the experiments, a constant step-size of time interval was taken as $\Delta t=0.01$ (sec) and the variance of a white Gaussian noise was $\sigma^2=1.0$. A set of parameter values in Fig.4.4 was preassigned as $\omega_1^2=1.0$, $\alpha=10$, $\varepsilon=0.01$ and $\delta=0.1$ and in Fig.4.5 as $\alpha=4$, $\varepsilon=0.02$ and $\delta=0.14$. Both of the initial values in Figs.4.4 and 4.5 were set as $r_0=2.0$, respectively. The solid line I represents the behavior of the deterministic system, which converges to the stationary state r=0 after t=50 sec and the solid line II shows a representative of sample runs of the stochastic systems. In Fig.4.4, the sample trajectory r(t) converges to I_1' with $r_e=\sqrt{\sigma^2/4\alpha}=0.16$ and, from Eq.(4.22), we have the summarized results,

$$(4.36)\quad P_r(r(t)\in[0,0.32]\,|\,r(0)=2.0) = 0.86$$

under the conditions $r_i=0$ and $r_j=0.32$ in Eq.(4.35). Similarly, Fig.4.5 shows a sample run which converges to $I_2'=[0,0.50]$ with $P_r=0.86$ where $r_e=0.25$. It is confirmed that both Figs.4.4 and 4.5 demonstrate the sample behaviors to show the validity of the theoretical results.

4.4.B Example-2

Let the nonlinear function $g(x,\dot{x})$ and $h(x,\dot{x})$ be given by

$$(4.37)\quad g(x,\dot{x}) \equiv g(x_1,x_2) = -(1-x_1^2)x_2$$

and

$$(4.38)\quad h(x,\dot{x}) \equiv h(x_1,x_2) = \gamma$$

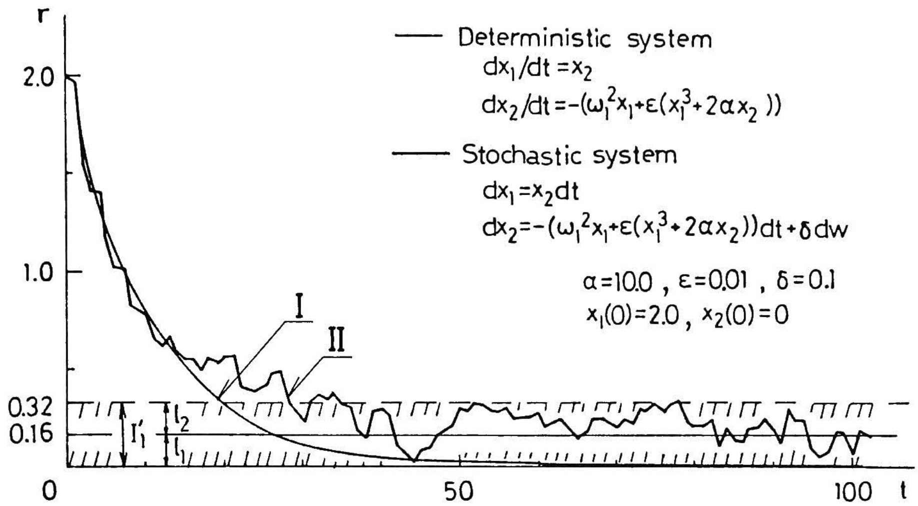


Fig.4.4 Sample Behavior of the System given by Eq.(4.27) (The Case of $I'_1 = [0, 0.32]$)

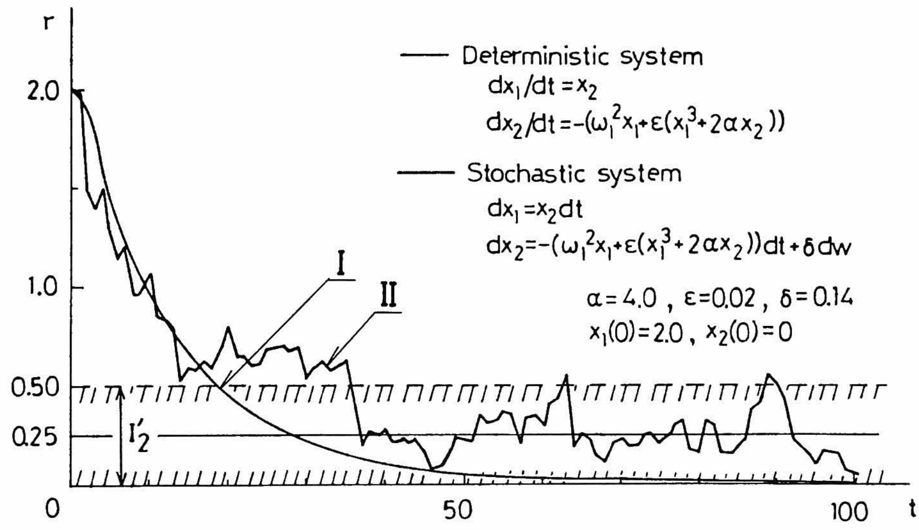


Fig.4.5 Sample Behavior of the System given by Eq.(4.27) (The Case of $I'_2 = [0, 0.50]$)

respectively, where γ is a constant. Then, we have

$$(4.39a)\quad dx_1 = x_2 dt$$

$$(4.39b)\quad dx_2 = -\{x_1 - \varepsilon(1 - x_1^2)x_2\}dt - \delta\gamma dw.$$

Equation (4.39) is a mathematical model of nonlinear dynamical systems of Van der Pol type when dw=0. Using Eqs.(4.2) and (4.3), the diffusion and drift terms are respectively computed to be

$$(4.40)\quad U^2(r) = \frac{\sigma^2\gamma^2}{4}$$

and

$$(4.41)\quad V(r) = \frac{\sigma^2\gamma^2}{4r} + \frac{r}{4}(2 - \frac{r^2}{2}) .$$

Since no singular points exist from (4.40), an application of Theorem 4.1 brings the following results.

(i) It is obvious that Eqs.(4.40) and (4.41) satisfy (C.1) and (C.2), respectively.

(ii) With the help of Eqs.(4.40) and (4.41), the calculations of (C.3) become

$$(4.42)\quad \lim_{r\to 0} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)}d\eta\} = \lim_{r\to 0} R_1 r\exp[\frac{4}{\sigma^2\gamma^2}\frac{r^2}{4}(1 - \frac{r}{6})\}] = 0$$

and furthermore

$$(4.43)\quad \lim_{r\to\infty} \exp\{\int_{r_a}^{r} \frac{V(\eta)}{U^2(\eta)}d\eta\} = 0,$$

where $R_1 = \exp\{r_a^2(r_a-6)/6\sigma^2\gamma^2\}/r_a$.

Equations (4.42) and (4.43) imply that there exists an invariant measure $\mu(A)$ in Eq.(4.39).

Similarly as in Example-1, we shall clarify the system behaviors by applying Theorem 4.2 to the system (4.39). From Eq.(4.41), it is obvious that $V(r_e)=0$ holds only when

$$(4.44) \quad r = r_e = [2 + \sqrt{4 + 2\sigma^2\gamma^2}]^{1/2}.$$

We shall consider the non-singular interval $I' = [r_i, r_j]$ including $r=r_e$ obtained by Eq.(4.44). It is obvious that I' satisfy (C.5) of Theorem 4.2. Furthermore, with the help of Eqs.(4.40) and (4.41), the computation of Eq.(4.22) becomes,

$$(4.45) \quad P_r(r(t)\epsilon I' | r(0)=r_0)$$

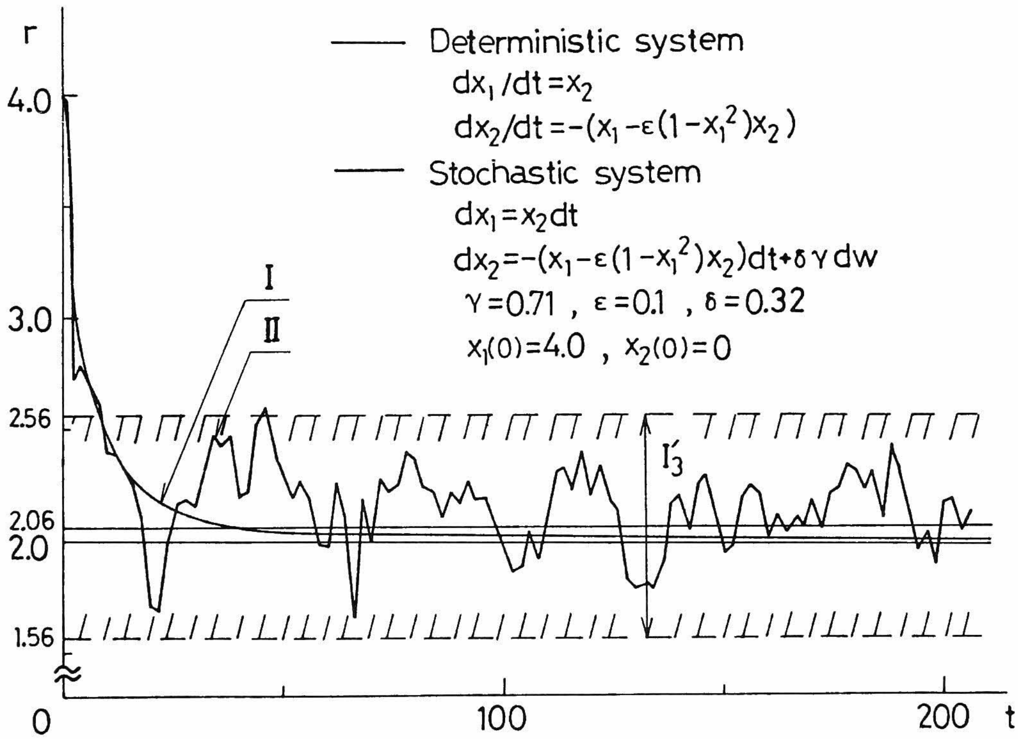


Fig.4.6 Sample Behavior of the System given by Eq.(4.39) (The Case of $I_3' = [1.56, 2.56]$)

$$= \int_{r_i}^{r_j} r\exp\{\frac{1}{\sigma^2\gamma^2}(r^2-\frac{r^4}{8})\}dr \Big/ \int_0^{\infty} r\exp\{\frac{1}{\sigma^2\gamma^2}(r^2-\frac{r^4}{8})\}dr$$

$$\cong \frac{\exp\{-(1-\frac{\sigma^2\gamma^2}{4})\exp(\frac{1}{\sigma^2\gamma^2}(r_i^2-4))\}-\exp\{-(1-\frac{\sigma^2\gamma^2}{4})\exp(\frac{1}{\sigma^2\gamma^2}(r_j^2-4))\}}{\exp\{-(1-\frac{\sigma^2\gamma^2}{4})\exp(-\frac{4}{\sigma^2\gamma^2})\}}.$$

A result of simulation experiments is shown in Fig.4.6. Parameters of Eq.(4.39) were set as $\sigma^2=1$, $\gamma=0.71$ and $\varepsilon=0.1$. The initial value was $r_0=4.0$. The solid line I shows the limit cycle with the radius $r=\sqrt{x_1^2 + x_2^2}=2.0$ in the stationary state after t=50 sec. The solid line II represents a sample trajectory of the r(t)-process which converges to the interval I_3' including a point $r=r_e=[2+\sqrt{4+2\sigma^2\gamma^2}]^{1/2}$ =2.06 with the probability $P_r=0.96$. This value is easily obtained by Eq.(4.45), letting $r_i=1.56$ and $r_j=2.56$.

Detailed aspects of stochastic behaviors of the nonlinear

Table 4.1 Comparison of Convergence Point between Deterministic and Stochastic Systems

System State / Examples	Convergence Point	
	Deterministic system	Stochastic system
Example 1 Eq.(4.27)	$r_c = 0$	$r_e= \sqrt{\sigma^2/4\alpha}$
Example 2 Eq.(4.39)	$r_c = 2.0$	$r_e= [2+\sqrt{4+2\sigma^2\gamma^2}]^{1/2}$

dynamical systems considered in Examples 1 and 2 are summarized in Table 4.1. In these two numerical examples, it is interested in observing that the stationary state of the deterministic dynamical system is shifted by the application of random excitations, for instance, in Example-1, the steady state $r_c = 0$ of the r(t)-process shifted into $r_e = \sqrt{\sigma^2/4\alpha}$ by the existence of random noise with the variance σ^2. The steady state in Example-2 is also shifted from $r_c=2.0$ for the deterministic system to $r_e = [2+\sqrt{4+2\sigma^2\gamma^2}]^{1/2}$ for the stochastic one.

4.5 Summary

In this chapter, new analytical approaches have been developed to explore the asymptotic behaviors of nonlinear stochastic dynamical systems with the differential generator of non-degenerate type. Two theorems were demonstrated giving sufficient conditions for the existence of the stationary response and for the convergence of sample trajectories to the stationary state with a certain probability appraisal. As the result, it was verified that the stationary state of the deterministic dynamical system is shifted by the application of random excitations. The validity of the methods presented here was shown through sample trajectories performed by digital simulation studies.

CHAPTER 5

STOCHASTIC STABILITY OF NONLINEAR DYNAMICAL SYSTEMS CONSIDERING INITIAL STATES

5.1 Introduction

One of remarkable features in behaviors of nonlinear dynamical systems is the system response to be dependent on initial conditions, which is an inherent characteristics due to the existence of nonlinearities. Then, the initial value as strong as nonlinear characteristics effects on the asymptotic stability of solution processes. In the version of nonlinear stochastic systems, based on the background knowledge of the stochastic process theory, there are some investigations [25],[31],[33],[71] of stochastic stability, considering only the nonlinear characteristics in their dynamics. However, none has established any new generalized method, which clarifies the effect of initial states on the system behavior. This is because the stability analysis until now has been concerned with the asymptotic stability for the sufficiently small initial state near the origin and then the general analytical approach has not yet been established to a global asymptotic stability for every initial states.

Now, let $x(t,r_0)$ be the solution process of the stochastic system starting at the initial state r_0, where $x(t,r_0)$ is the n-dimensional state vector and where $x(t,0)\equiv 0$ an equilibrium point. If the solution process $x(t,r_0)$ starting at the neighborhood of the equilibrium point takes a finite value for all t, the equilibrium solution is said to be stable. The definition of the asymptotic stability is stated as follows:

[Definition 5.1] (Stochastically asymptotic stability)

The equilibrium solution is said to be asymptotically stable with probability more than ε_0, if the equilibrium solution is stable and, for $\varepsilon_0=\varepsilon_0(r_0)>0$,

$$P\{ \lim_{t\to\infty} \sup \| x(t,r_0) \| = 0 \} \geq \varepsilon_0, \quad \varepsilon_0 = 1 \text{ as } r_0 \to 0$$

holds. Here, $\| \cdot \|$ denotes the absolute value norm.

[Definition 5.2] (Stochastically asymptotic stability in the large)

The equilibrium solution is said to be asymptotically stable in the large, if the equilibrium solution is stable and, for every initial value $r_0 \epsilon E^{(n)}$,

$$P\{ \lim_{t\to\infty} \sup \| x(t,r_0) \| = 0 \} = 1$$

holds.

The concept of stochastic stability described above is shown in Fig.5.1. Now, we shall consider trajectories of the x(t)-process starting at the initial state r_0. The x(t)-processes starting at r_{01} show the case that the behaviors are dependent on

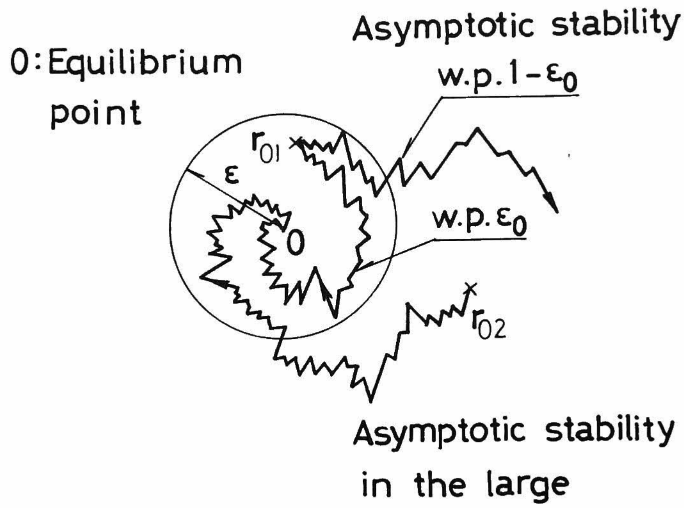


Fig.5.1 Concept of Asymptotic Stability and Asymptotic Stability in the Large

the initial states by effects of the nonlinear characteristics. Accordingly, the x(t)-processes are asymptotically stable with probability ε_0. On the other hand, the x(t)-process originating at r_{02} is asymptotically stable in the large which reaches to the equilibrium point w.p.1 for every initial states.

This chapter is concerned with a realizable approach to solve stochastically the asymptotic stability for nonlinear systems with (1) a random parameter modeled by a white Gaussian random process and (2) two random parameters modeled by a white Gaussian and a finite state Markov chain processes respectively. Figure 5.2 illustrates the analytical procedure which will be developed in the sequel. In the former case, for the purpose of finding the differential generator of the r(t)-process, the averaging principle by

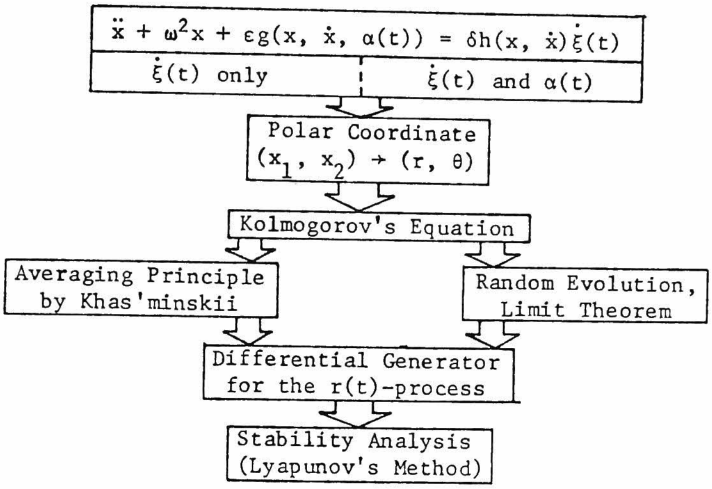


Fig.5.2 Orientation of Analytical Procedure

Khas'minskii is applied to the Kolmogorov equation derived by using the polar coordinate transformation. On the other hand, in the latter case, the concept of random evolutions is introduced instead of the averaging principle because the system involves the parameter of Markov chain type. The system stability is finally examined by using the differential generator in terms of stochastic behaviors of the $r(t)$-process.

A stochastic Lyapunov function approach to explore the asymptotic stability is demonstrated in Sections 5.2 and 5.3, taking into account the influence of the initial conditions. In Sections 5.4 and 5.5, a general class of nonlinear dynamical systems with two random parameters is considered. For the purpose of examining the asymptotic behavior, the concept of random evolution is introduced. Section 5.6 is devoted to demonstrating illustrative examples.

5.2 Differential Generator Associated with Basic Equation

We shall consider a nonlinear dynamical system modeled by

$$(5.1)\qquad \ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = \delta h(x,\dot{x})\dot{\xi}(t)$$

with the given initial values $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$, where ε and δ are small constants, g and h nonlinear functions respectively and $\dot{\xi}(t)$ a white Gaussian noise and where "$\cdot$" expresses the differentiation with respect to time t. Equation (5.1) may be considered as a generalization of mathematical models of dynamical systems such that the system is lightly damped, weakly nonlinear and that the system response is related to a random excitation with relatively small magnitude [51],[52].

With $x=x_1$, $\dot{x}=x_2$, Eq.(5.1) is expressed by the following stochastic differential equation of Itô-type [79],

$$(5.2a)\qquad dx_1 = x_2 dt$$

$$(5.2b)\qquad dx_2 = -\{\omega^2 x_1 + \varepsilon g(x_1,x_2)\}dt + \delta h(x_1,x_2)dw(t)$$

where the w(t)-process is the Brownian motion process with the following properties; $E_b\{dw(t)\}=0$ and $E_b\{(dw(t))^2\}=\sigma^2 dt$, where σ is a constant.

It can easily be expected that the two-dimensional dynamical system given by (5.2) is converted into the one-dimensional system along the relation,

$$(5.3)\qquad x_1 = \frac{r}{\omega}\sin\theta, \qquad x_2 = r\cos\theta.$$

Naturally, the converted one-dimensional process r(t) is

$$(5.4)\qquad r^2(t) = \omega^2 x_1^2 + x_2^2.$$

After somewhat tedious calculations using the averaging principle[80], it may be found that the r(t)-process is Markovian with the differential generator (see Chapter 4),

$$(5.5) \qquad L_r = U^2(r)\frac{d^2}{dr^2} + V(r)\frac{d}{dr} ,$$

where

$$(5.6) \qquad U^2(r) = \frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\sin\theta, r\cos\theta)\cos^2\theta d\theta ,$$

$$(5.7) \qquad V(r) = \frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\sin\theta, r\cos\theta)\frac{\sin^2\theta}{r}d\theta$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} g(\frac{r}{\omega}\sin\theta, r\cos\theta)\cos\theta d\theta$$

and $\psi-\omega t = \theta+\pi/2$, and where it has been assumed that $\delta^2=\varepsilon$.

5.3 A New Lyapunov Function

We need the following lemma associated with the asymptotic stability criteria of the r(t)-process.

[Lemma 5.1] For a fixed m, assume the following conditions (A.1) to (A.3):

(A.1) $W_L(r)$ is non-negative and continuous in the open set $Q_m \triangleq \{ r ; W_L(r)<m \}$.

(A.2) r(t) is a right continuous strong Markov process with the weak infinitesimal operator $\tilde{A}_m$ defined in Q_m.

(A.3) $\tilde{A}_m W_L(r) = -k(r) \le 0$.

Letting $R_m = Q_m \cap \{ r ; k(r)=0 \}$, then

$$(5.8) \qquad P_0\{ \lim_{t\to\infty} r(t)\epsilon R_m \} \geq 1 - \frac{W_L(r_0)}{m} ,$$

where $P_0\{\cdot\}$ is the probability of $\{\cdot\}$, provided that the $r(t)$-process starts with $r(0) = r_0$ at the initial time $t = 0$.

Furthermore, assume that the assumptions (A.1) to (A.3) hold for $^\forall m>0$. Letting $R \triangleq \cup_{m=1}^{\infty} R_m$, then

$$(5.9) \qquad P_0\{ \lim_{t\to\infty} r(t)\epsilon R \} = 1.$$

Since the proof is straightforward by using the super-martingale property of $W_L(r)$, we shall omit to write here.

The following theorem gives sufficient conditions of the asymptotic stability with probability one.

[Theorem 5.1] For an arbitrarily fixed initial value $r(0)=r_0$, assume that the coefficients of L_r in (5.5) satisfy the following conditions:

(C.1) $U^2(r) = 0$ if and only if $r = 0$, and $V(0) = 0$.

(C.2) $\lim_{r\to 0} |\frac{V(r)}{U^2(r)}| \exp\{\int_r^{r_0} \frac{V(\zeta)}{U^2(\zeta)} d\zeta\} < \infty.$

(C.3) $\int_{r_0}^{\infty} \frac{V(\zeta)}{U^2(\zeta)} d\zeta < \infty.$

Then, for any initial value $r_0 \epsilon [0,\infty)$, the following equality holds:

$$(5.10) \qquad P_0\{ \lim_{t\to\infty} r(t)=0 \} = 1.$$

(Proof) Let $\psi(t)$ be an arbitrary positive smooth function such that

$$(5.11)\qquad \int_0^\infty \psi(r)dr < \infty .$$

Define $W_L(r)$ by

$$(5.12)\qquad W_L(r) \triangleq \int_0^r \exp\{ \int_\eta^{r_0} (\frac{V(\zeta)}{U^2(\zeta)} + \psi(\zeta))d\zeta \}d\eta.$$

From (5.12), it follows that

$$(5.13)\qquad \frac{dW_L(r)}{dr} = \exp\{ \int_r^{r_0} (\frac{V(\zeta)}{U^2(\zeta)} + \psi(\zeta))d\zeta\}.$$

and

$$(5.14)\qquad \frac{d^2W_L(r)}{dr^2} = -[\frac{V(r)}{U^2(r)} + \psi(r)]\exp\{ \int_r^{r_0} (\frac{V(\zeta)}{U^2(\zeta)} + \psi(\zeta))d\zeta\}.$$

Hence, it can easily be examined that

$$(5.15)\qquad L_rW_L(r) = -U^2(r)\psi(r)\exp\{ \int_r^{r_0} (\frac{V(\zeta)}{U^2(\zeta)} + \psi(\zeta))d\zeta\}$$

$$\leq 0.$$

The assumptions (A.1) to (A.3) are thus satisfied and $W_L(r)$ becomes a Lyapunov function, if the following conditions (c.i) and (c.ii) are satisfied:

(c.i) $W_L(r) \to \infty$ as $r \to \infty$.

(c.ii) For any bounded r,

$$W_L(r) < \infty, \qquad \frac{dW_L(r)}{dr} < \infty, \qquad |\frac{d^2W_L(r)}{dr^2}| < \infty.$$

With the definition (5.12), the first condition (c.i) is

$$(5.16)\qquad \lim_{r\to\infty} \exp\{ \int_r^{r_0} (\frac{V(\zeta)}{U^2(\zeta)} + \psi(\zeta))d\zeta\} \neq 0$$

or equivalently

$$(5.17)\qquad \int_{r_0}^{\infty} \left(\frac{V(\zeta)}{U^2(\zeta)} + \psi(\zeta)\right)d\zeta < \infty.$$

From (C.3), (5.11) and (5.17), the condition (c.i) holds.

Since it is apparent that, if $|d^2W_L(r)/dr^2| < \infty$, then $W_L(r) < \infty$ and $dW_L(r)/dr < \infty$, it is sufficient to show that the third inequality of the condition (c.ii) holds. From (C.1), the origin r=0 is the only one singular point. Consequently, with the conditions (C.1) and (C.2) and the inequality (5.11), we have

$$(5.18)\qquad \lim_{r\to 0} \left|\frac{V(r)}{U^2(r)} + \psi(r)\right| \exp\{ \int_{r}^{r_0} \left(\frac{V(\zeta)}{U^2(\zeta)} + \psi(\zeta)\right)d\zeta\} < \infty,$$

from which the condition (c.ii) holds. Thus the proof has been completed.

We shall proceed to state sufficient conditions for the asymptotic stability with the probability appraisal.

[Theorem 5.2] Assume that the following condition holds together with the conditions (C.1) and (C.2).

(C.4) There exists a positive constant M such that, for any $r\in(0,M)$, the drift term V(r) given by (5.7) is negative.

Then, we have

$$(5.19)\qquad P_0 \{\lim_{t\to\infty} r(t)=0\} \geq 1 - \frac{W_L(r_0)}{W_L(M)} .$$

(Proof) Define

$$(5.20)\qquad Q_m' \triangleq \{ r ; r\in[0,M)\} \equiv \{ r ; W_L(r)<m\},$$

where $m=W_L(M)$. Furthermore, for $r\in Q_m'$, let $\phi(r)$ be an arbitrary smooth and positive function related to the function $\Psi(r)$ by

$$(5.21) \qquad \Psi(r) \triangleq -V(r)\phi(r),$$

where

$$(5.22) \qquad \int_0^M \{-V(r)\phi(r)\}dr < \infty.$$

From (5.12) and (5.15), the Lyapunov function $W_L(r)$ and its differential generator are respectively expressed by

$$(5.23) \qquad W_L(r) = \int_0^r \exp\{ \int_\eta^{r_0} (\frac{V(\zeta)}{U^2(\zeta)} - V(\zeta)\phi(\zeta))d\zeta \}d\eta$$

and

$$(5.24) \qquad L_r W_L(r) = V(r)U^2(r)\phi(r)\exp\{ \int_r^{r_0} (\frac{V(\zeta)}{U^2(\zeta)} - V(\zeta)\phi(\zeta))d\zeta \}$$

$$\leq 0$$

for $r,\ r_0 \in Q_m'$.

We shall examine the conditions (c.i) and (c.ii) in the proof of Theorem 5.1. By the conditions (C.1) and (C.2), it is obvious that the condition (c.ii) holds. Furthermore, it is a direct consequence from (5.23) that $W_L(r)$ is monotone increasing with respect to r in Q_m'. Consequently, using the inequality (5.8), the asymptotic stability is concluded with the probability appraisal $1-W_L(r_0)/W_L(M)$.

5.4 Extension to Dynamical Systems with Random Coefficients

In this section, an extension of the results obtained in the previous section is demonstrated to the dynamical system modeled by

$$(5.25) \qquad \ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x},\alpha(t)) = \delta h(x,\dot{x})\dot{\xi}(t)$$

with the given initial conditions, $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$, where $\alpha(t)$ is a parametric noise process expressed mathematically by an ergodic Markov chain with finite n-stages and $\alpha(0)=\alpha_i (i=1,2,\cdots,n)$. Equation (5.25) may be considered as a mathematical model of a class of lightly damped nonlinear dynamical systems excited by a random input, whose parameter changes with time taking n modes according to a continuous-time Markov chain with the infinitesimal generator Q.

The stochastic differential equation of Itô-type associated with Eq.(5.25) is easily derived as

$$(5.26a)\quad dx_1 = x_2 dt$$

$$(5.26b)\quad dx_2 = -\{\omega^2 x_1 + \varepsilon g(x_1,x_2,\alpha(t))\}dt + \sqrt{\varepsilon}h(x_1,x_2)dw(t)$$

where, for convenience of discussions, we set as $\delta=\sqrt{\varepsilon}$.

Noting that a Markov chain process may be regarded as a special class of Poisson processes, it can easily be understood that the joint process $(x_1,x_2,\alpha(t))$ is a pair Markov process perturbed randomly by both the Brownian motion and Poisson processes[81],[82]. Hence, defining the probability density of a transition from the state $X_i=(x_1,x_2,\alpha_i)$ to another state $Y_j=(y_1,y_2,\alpha_j)$ by $p_i=p_i(X_i;t;Y_j)$, the probability density p_i for the fixed Y_j satisfies

$$(5.27)\quad \frac{\partial p_i}{\partial t} = -x_2\frac{\partial p_i}{\partial x_1} - \{\omega^2 x_1 + \varepsilon g(x_1,x_2,\alpha_i)\}\frac{\partial p_i}{\partial x_2} + \frac{\varepsilon}{2}\sigma^2 h^2(x_1,x_2)\frac{\partial^2 p_i}{\partial x_2^2} + \sum_{k=1}^{n} q_{ik}p_k$$

with the initial condition $p_i(X_i;0;Y_j)=\delta_{ij}\delta(x_1-y_1,x_2-y_2)$, where

q_{ik} is the (i,k)th element of an n×n matrix Q, δ_{ij} the Kronecker delta and δ the Dirac delta function. It is well-known that

$$(5.28a)\quad \lim_{\Delta t\to 0} P_r\{\alpha(t+\Delta t)=\alpha_k|\alpha(t)=\alpha_i\} = q_{ik}\Delta t + o(\Delta t)$$

and

$$(5.28b)\quad \lim_{\Delta t\to 0} P_r\{\alpha(t+\Delta t)=\alpha_i|\alpha(t)=\alpha_i\} = 1 + q_{ii}\Delta t + o(\Delta t),$$

where $q_{ik}\geq 0$ for $i\neq k$, $q_{ii}\leq 0$ and $\Sigma_{k=1}^{n} q_{ik}=0$ for $i=1,2,\cdots,n$.

We shall convert the $(x_1,x_2,\alpha(t))$-process into the $(r,\theta,\alpha(t))$-process along the relation,

$$(5.29a)\quad x_1 = \frac{r}{\omega}\sin\theta, \quad x_2 = r\cos\theta$$

and

$$(5.29b)\quad r^2 = \omega^2 x_1^2 + x_2^2 .$$

Noting that the zero solution $x_1=x_2=0$ to Eq.(5.26) implies $r=0$ which is a reflecting barrier, the $r(t)$-process may be considered within the semi-infinite interval $r\epsilon[0,\infty)$. With the relation (5.29), we write $v(r,\theta,\alpha_i;t;r_1,\theta_1,\alpha_j)$ for $p_i(x_1,x_2,\alpha_i;t;y_1,y_2,\alpha_j)$ and abbreviate it by $v_i(r,\theta,t)$ for a set of fixed values, r_1,θ_1 and α_j. Letting $\theta=\phi-\omega t$, then, after somewhat tedious caluculations, we have

$$(5.30)\quad \frac{\partial v_i}{\partial t} = \varepsilon\{\frac{\sigma^2}{2}h^2[\cos^2(\phi-\omega t)\frac{\partial^2 v_i}{\partial r^2} - \frac{\sin 2(\phi-\omega t)}{r}\frac{\partial^2 v_i}{\partial r\partial\phi} + \frac{\sin^2(\phi-\omega t)}{r^2} \times\frac{\partial^2 v_i}{\partial\phi^2} + \frac{\sin 2(\phi-\omega t)}{r^2}\frac{\partial v_i}{\partial\phi} + \frac{\sin^2(\phi-\omega t)}{r}\frac{\partial v_i}{\partial r}] + g(\alpha_i) \times\frac{\sin(\phi-\omega t)}{r}\frac{\partial v_i}{\partial\phi} - g(\alpha_i)\cos(\phi-\omega t)\frac{\partial v_i}{\partial r}\} + \sum_{k=1}^{n} q_{ik}v_k,$$

$$(i = 1,2,\cdots,n)$$

with the initial condition $v_i(r,\phi,0)=\delta_{ij}\delta(r-r_1,\phi-\phi_1)$, where $g(\alpha_i)=g\{r\sin(\phi-\omega t)/\omega, r\cos(\phi-\omega t),\alpha_i\}$ and $\phi_1=\theta_1$. The limiting behavior of $v_i(t)$ is investigated by tending ε to zero and t to infinitive under the condition that εt is constant. To do this, changing the time scale t for τ and writing $v_i^{(\varepsilon)}(\tau)$ for $v_i(\tau/\varepsilon)$, where $\tau=\varepsilon t$, Eq.(5.30) is written by

$$(5.31)\quad \frac{\partial v_i^{(\varepsilon)}}{\partial\tau} = \frac{\sigma^2}{2}h^2[\cos^2(\phi-\frac{\omega\tau}{\varepsilon})\frac{\partial^2 v_i^{(\varepsilon)}}{\partial r^2} - \frac{\sin2(\phi-\frac{\omega\tau}{\varepsilon})}{r}\frac{\partial^2 v_i^{(\varepsilon)}}{\partial r\partial\phi}$$

$$+ \frac{\sin^2(\phi-\frac{\omega\tau}{\varepsilon})}{r^2}\frac{\partial^2 v_i^{(\varepsilon)}}{\partial\phi^2} + \frac{\sin2(\phi-\frac{\omega\tau}{\varepsilon})}{r^2}\frac{\partial v_i^{(\varepsilon)}}{\partial\phi} + \frac{\sin^2(\phi-\frac{\omega\tau}{\varepsilon})}{r}$$

$$\times\frac{\partial v_i^{(\varepsilon)}}{\partial r} + g(\alpha_i)\frac{\sin(\phi-\frac{\omega\tau}{\varepsilon})}{r}\frac{\partial v_i^{(\varepsilon)}}{\partial\phi} - g(\alpha_i)\cos(\phi-\frac{\omega\tau}{\varepsilon})\frac{\partial v_i^{(\varepsilon)}}{\partial r}$$

$$+ \frac{1}{\varepsilon}\sum_{k=1}^{n} q_{ik}v_k^{(\varepsilon)}, \qquad (i=1,2,\cdots,n).$$

Let $A_i(r,\phi-\frac{\omega\tau}{\varepsilon})$ be the differential generator of the right hand side in Eq.(5.31). Equation (5.31) is written by

$$(5.32)\quad \frac{\partial v_i^{(\varepsilon)}}{\partial\tau} = A_i v_i^{(\varepsilon)} + \frac{1}{\varepsilon}\sum_{k=1}^{n} q_{ik}v_k^{(\varepsilon)}, \qquad (i=1,2,\cdots,n).$$

Assume that the value of ε is sufficiently small and, for any fixed $\tau>0$, define $L_{\alpha i}$ by

$$(5.33)\quad L_{\alpha i} \triangleq \lim_{T\to\infty}\frac{1}{T}\int_0^T A_i(r,\phi-\frac{\omega\tau}{\varepsilon})d(\frac{\tau}{\varepsilon}).$$

Noting that $A_i(r,\phi-\frac{\omega\tau}{\varepsilon})$ is periodic with respect to $\phi-(\omega\tau/\varepsilon)=\psi$, then, from (5.33), $L_{\alpha i}$ is computed to be

$$(5.34)\quad L_{\alpha i} = \frac{1}{2\pi}\int_0^{2\pi} A_i(r,\psi)d\psi$$

$$= \frac{\sigma^2}{4\pi}[(\int_0^{2\pi} h^2\cos^2\psi d\psi)\frac{\partial^2}{\partial r^2} - (\int_0^{2\pi} h^2\frac{\sin 2\psi}{r}d\psi)\frac{\partial^2}{\partial r\partial\phi}$$

$$+ (\int_0^{2\pi} h^2\frac{\sin^2\psi}{r^2}d\psi)\frac{\partial^2}{\partial\phi^2} + (\int_0^{2\pi} h^2\frac{\sin 2\psi}{r^2}d\psi)\frac{\partial}{\partial\psi} + (\int_0^{2\pi} h^2\frac{\sin^2\psi}{r}d\psi)$$

$$\times\frac{\partial}{\partial r}] + (\frac{1}{2\pi}\int_0^{2\pi} g(\alpha_i)\frac{\sin\psi}{r}d\psi)\frac{\partial}{\partial\phi} - (\frac{1}{2\pi}\int_0^{2\pi} g(\alpha_i)\cos\psi d\psi)\frac{\partial}{\partial r}.$$

Although Eq.(5.34) plays a basic role to explore stochastic behaviors of the nonlinear dynamical system (5.25), our attention is focussed on the asymptotic aspect of Eq.(5.26) with an averaged differential generator rather than $v_i^{(\varepsilon)}$ themselves. Now, let $\overline{P}_i$ be

$$(5.35)\quad \overline{P}_i = \lim_{t\to\infty} P_r\{\alpha(t)=\alpha_i\}$$

for any $\alpha(0)=\alpha_j$, where $\sum_i\overline{P}_i=1$. Furthermore, define

$$(5.36)\quad \overline{\alpha} = \sum_{i=1}^{n} \overline{P}_i\alpha_i$$

and

$$(5.37)\quad \overline{L} = \sum_{i=1}^{n} \overline{P}_i L_{\alpha i}.$$

Then, from (5.34), the averaged differential generator $\overline{L}$ can be obtained by

$$(5.38)\quad \overline{L} = (\frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2\cos^2\psi d\psi)\frac{\partial^2}{\partial r^2} - (\frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2\frac{\sin\psi}{r}d\psi)\frac{\partial^2}{\partial r\partial\psi}$$

$$+ \left(\frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2 \frac{\sin^2\psi}{r^2} d\psi\right)\frac{\partial^2}{\partial\phi^2} + \left(\frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2 \frac{\sin 2\psi}{r^2} d\psi + \frac{1}{2\pi}\int_0^{2\pi} g(\bar{\alpha})\frac{\sin\psi}{r}\right.$$

$$\left.\times d\psi\right)\frac{\partial}{\partial\phi} + \left(\frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2 \frac{\sin^2\psi}{r} d\psi - \frac{1}{2\pi}\int_0^{2\pi} g(\bar{\alpha})\cos\psi d\psi\right)\frac{\partial}{\partial r} \, .$$

(for more details, see Reference [97]).

5.5 Asymptotic Stability Theorems

In this section, asymptotic behaviors of the stochastic nonlinear system given by Eq.(5.25) are examined. Our main concern is the asymptotic behavior of the zero solution $x_1=x_2=0$ which implies $r=0$. Our attention is thus directed to the $r(t)$-process whose differential generator is given by (5.38). However, if there exists a stationary density $p(r,\phi)$ for the process described by (5.38), then it will apparently be independent of ϕ. This fact allows us to write (5.38) in a simpler form as

$$(5.39) \qquad \bar{L}_r = U^2(r)\frac{\partial^2}{\partial r^2} + V_r(r,\bar{\alpha})\frac{\partial}{\partial r}$$

where

$$(5.40) \qquad V_r(r,\bar{\alpha}) = \frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2\left(\frac{r}{\omega}\sin\psi, r\cos\psi\right)\frac{\sin^2\psi}{r} d\psi$$

$$- \frac{1}{2\pi}\int_0^{2\pi} g\left(\frac{r}{\omega}\sin\psi, r\cos\psi, \bar{\alpha}\right)\cos\psi d\psi \, .$$

It can thus be understood that theoretical considerations run on the same line as described in Section 5.3. The following theorem gives sufficient conditions for the asymptotic stability in the large.

[Theorem 5.3] Assume that, for any fixed initial value $r(0)=r_0$, the coefficients $U^2(r)$ and $V_r(r,\bar{\alpha})$ in (5.39) satisfy the following conditions:

(C.5) $U^2(r)=0$, if and only if $r=0$ and $V_r(0,\bar{\alpha})=0$.

(C.6) $\lim_{r\to 0} \left|\frac{V_r(r,\bar{\alpha})}{U^2(r)}\right| \exp\{\int_r^{r_0} \frac{V_r(\zeta,\bar{\alpha})}{U^2(\zeta)}d\zeta\} < \infty.$

(C.7) $\int_{r_0}^{\infty} \frac{V_r(\zeta,\bar{\alpha})}{U^2(\zeta)}d\zeta < \infty.$

Then, for any initial value $r_0\in[0,\infty)$ and $\alpha_i (i=1,2,\cdots,n)$, we have

$$(5.41)\quad P_r\{\lim_{t\to\infty} r(t)=0|r(0)=r_0,\ \alpha(0)=\alpha_i\} = 1.$$

The following theorem gives also sufficient conditions with the probability appraisal.

[Theorem 5.4] Assume that the following conditions are satisfied together with the conditions (C.5) and (C.6):

(C.8) There exists a positive constant M such that, for any $r\in(0,M)$, the drift term $V_r(r,\bar{\alpha})$ in (5.39) is negative.

(C.9) The initial value $\alpha(0)=\alpha_i$ satisfies that, for any fixed r, $V_r(r,\bar{\alpha})>V_r(r,\alpha_i)$.

Then, we have

$$(5.42)\quad P_r\{\lim_{t\to\infty} r(t)=0|r(0)=r_0<M,\ \alpha(0)=\alpha_i\} \geq 1 - \frac{W_\alpha(r_0)}{W_\alpha(M)},$$

where

$$(5.43)\quad W_\alpha(r) = \int_0^r \exp\{\int_\eta^{r_0} (\frac{V_r(\zeta,\bar{\alpha})}{U^2(\zeta)} - V_r(\zeta,\bar{\alpha})\phi(\zeta))d\zeta\}d\eta.$$

Since proofs of Theorems 5.3 and 5.4 are essentially the same as those of Theorems 5.1 and 5.2 except for the condition (C.9), descriptions are omitted here. For the condition (C.9), in the neighborhood of $t=0$, it is necessary to assume that the $r(t)$-process does not go out of the domain $0 \le r \le M$. To do this, bearing in mind the fact that the initial value $\alpha(0)=\alpha_i$ is considered so as to satisfy

$$(5.44) \qquad u^{(0)}(0) = \sum_{i=1}^{n} \overline{P}_i f(\alpha_i),$$

where

$$(5.45) \qquad f(\alpha_i) = u_i(r,\theta,\alpha_i;0;r_1,\theta_1,\alpha_j)$$

the initial value $\alpha(0)=\alpha_i$ lies on the domain such that the inequality $V_r(r,\overline{\alpha})>V_r(r,\alpha_i)$ holds for any fixed r.

5.6 Illustrative Examples

5.6.A Example-1

We shall consider a nonlinear dynamical system given by

$$(5.46) \qquad \ddot{x} + \{2\varepsilon\beta - \delta\dot{\xi}(t)\}\dot{x} + \omega^2 x + \varepsilon x^3 = 0.$$

Equation (5.46) may be considered as a mathematical model of dynamical systems whose damping coefficient is white Gaussian and restoring force is a nonlinearity of the cubic order. From Eq.(5.46), both the nonlinear functions g and h are respectively identified by

$$(5.47) \qquad g(x,\dot{x}) \equiv g(x_1,x_2) = x_1^3 + 2\beta x_2$$

and

$$(5.48) \qquad h(x,\dot{x}) \equiv h(x_1,x_2) = x_2 .$$

The precise interpretation of Eq.(5.46) is made by the following stochastic differential equation of Itô-type;

(5.49a) $dx_1 = x_2 dt$

(5.49b) $dx_2 = -\{\omega^2 x_1 + \varepsilon(2\beta x_2 + x_1^3)\}dt + \delta x_2 dw(t).$

From (5.6) and (5.7), the diffusion and drift coefficients are respectively computed to be

(5.50) $U^2(r) = \frac{3\sigma^2 r^2}{16}$

and

(5.51) $V(r) = (\frac{\sigma^2}{16} - \beta)r$.

Using (5.50) and (5.51), the conditions (C.1) to (C.3) in Theorem 5.1 are examined as follows:

(1) The condition (C.1) is obviously satisfied.

(2) The condition (C.2) holds for $\beta>\sigma^2/4$.

(3) The condition (C.3) holds for $\beta>\sigma^2/16$.

Consequently, we may conclude that the origin of the system (5.49) is asymptotically stable in the large under the condition

(5.52) $\beta > \sigma^2/4$.

On the other hand, choose the function $W_L(x_1,x_2)$ as [83]

(5.53) $W_L(x_1,x_2) = x_2^2 + 2\int_0^{x_1}(\omega^2 y + \varepsilon y^3)dy$.

Let L be the differential generator of Eq.(5.49). Then, from the relation,

(5.54) $LW_L = x_2\frac{\partial W_L}{\partial x_1} - \{\omega^2 x_1 + \varepsilon(2\beta x_2 + x_1^3)\}\frac{\partial W_L}{\partial x_2} + \frac{\delta^2\sigma^2 x_2^2}{2}\frac{\partial^2 W_L}{\partial x_2^2}$

$$= -x_2^2(4\varepsilon\beta - \delta^2\sigma^2)$$

and the assumption, $\delta^2=\varepsilon$, it follows that, for $\beta>\sigma^2/4$,

(5.55) $\quad LW_L \leq 0.$

It is obvious that the function W_L is the Lyapunov function and that

(5.56) $\quad P_r\{ \lim_{t\to\infty} x_2(t) = 0 \} = 1.$

Since the result (5.56) brings $P_r\{ \lim_{t\to\infty} x_1(t)=0 \}=1$[83], for any initial value r_0, we have the summarized result:

(5.57) $\quad P_r\{ \lim_{t\to\infty} x_1(t)=x_2(t)=0 \} = 1$

under the condition given by (5.52).

5.6.B Example-2

We shall consider a nonlinear dynamical system with a random coefficient given by

(5.58) $\quad \ddot{x} + \{2\varepsilon\alpha(t) - \delta\dot{\xi}(t)\}\dot{x} + \omega^2 x + \varepsilon x^3 = 0,$

where the parameter $\alpha(t)$ is considered to be a Markov chain discussed in the previous section. From Eq.(5.58), the stochastic differential equation of Itô-type becomes

(5.59a) $\quad dx_1 = x_2 dt$

(5.59b) $\quad dx_2 = -[\omega^2 x_1 + \varepsilon\{2\alpha(t)x_2 + x_1^3\}]dt + \delta x_2 dw(t).$

Bearing the relation (5.4) in mind and using (5.6) and (5.40), it follows that

(5.60) $U^2(r) = \frac{3}{16}\sigma^2 r^2$

and

(5.61) $V_r(r,\bar{\alpha}) = (\frac{\sigma^2}{16} - \bar{\alpha})r.$

From (5.60) and (5.61), the conditions (C.5) to (C.7) in Theorem 5.3 are examined as follows:

(i) The condition (C.5) is satisfied.

(ii) It can easily be seen that the condition (C.6) holds for $\bar{\alpha}>\sigma^2/4$.

(iii) For $\bar{\alpha}>\sigma^2/16$, the condition (C.7) is satisfied.

Hence, from Theorem 5.3, the equality (5.41) holds for any initial values r_0 and $\alpha(0)=\alpha_i$ under the condition that $\bar{\alpha}>\sigma^2/4$.

5.6.C Example-3

Consider a dynamical system modeled by the nonlinear differential equation of Rayleigh's type:

(5.62) $\ddot{x} + x + \varepsilon(1 - \dot{x}^2)\dot{x} = \delta x\dot{\xi}(t),$

where $x(0)=x_0$ and $\dot{x}(0)=\dot{x}_0$ as usual. It is well-known that, if $\delta=0$, then the system exhibits an unstable limit cycle and is asymptotically stable with respect to the origin.

Converting Eq.(5.62) into

(5.63a) $dx_1 = x_2 dt$

(5.63b) $dx_2 = -\{x_1 + \varepsilon(1 - x_2^2)x_2\}dt + \delta x_1 dw(t),$

and letting $r^2=x_1^2 + x_2^2$, the r(t)-process is the scalar Markov process whose diffusion and drift coefficients are respectively computed to be

$$(5.64)\qquad U^2(r) = \frac{\sigma^2}{16}r^2$$

and

$$(5.65)\qquad V(r) = \frac{3\sigma^2}{16}r - \frac{r}{2}\left(1 - \frac{3r^2}{4}\right).$$

By using (5.64) and (5.65), the conditions (C.1), (C.2) in Theorem 5.1 and (C.4) in Theorem 5.2 are examined as follows:

(i) The condition (C.1) holds.

(ii) The condition (C.2) holds, provided that $\sigma^2<2$.

(iii) From (5.65), since V(r) is negative for $r<\sqrt{(8-3\sigma^2)/6}$, the domain satisfying the condition (C.4) is

$$(5.66)\qquad Q_m' = \{\ r\ ;\ r < M=\sqrt{(8-3\sigma^2)/6}\ \}.$$

Thus, from Theorem 5.2, it may be concluded that, for the r(t)-process initiating at $r_0 \in Q_m'$,

$$(5.67)\qquad P_0\{\ \lim_{t\to\infty} r(t)=0\ \} \geq 1 - W_L(r_0)\ /\ W_L\left(\sqrt{\frac{8-3\sigma^2}{6}}\right).$$

5.6.D Example-4

The same system as in Example-3 is considered, besides the system parameter is modeled by a Markov chain, i.e.,

$$(5.68)\qquad \ddot{x} + x + \varepsilon\{1 - \alpha(t)\dot{x}^2\}\dot{x} = \delta x\dot{\xi}(t),$$

where $\alpha(0)=\alpha_i\,(i=1,2,\cdots,n)$. Equation (5.68) is converted into

$$(5.69a)\qquad dx_1 = x_2dt$$

$$(5.69b)\qquad dx_2 = -[x_1 + \varepsilon\{1 - \alpha(t)x_2^2\}x_2]dt + \delta x_1dw(t).$$

Hence, from (5.40), we have

$$(5.70)\qquad V_r(r,\bar{\alpha}) = \frac{\bar{\alpha}}{8} r\,(r^2 - \frac{8-3\sigma^2}{6\bar{\alpha}}).$$

By examining the conditions (C.5),(C.6),(C.8) and (C.9) and using Theorem 5.4, the sufficient condition for the asymptotic stability is found to be $\sigma^2<2$. Furthermore, it can easily be found that

$$(5.71)\qquad Q'_m = \{\ r\ ;\ r < M = \sqrt{\frac{8-3\sigma^2}{6\bar{\alpha}}}\ \},$$

where we assumed that $\bar{\alpha} > 0$, because the system is easily shown to be asymptotically stable in the large, if $\bar{\alpha}=0$.

The condition (C.9) shows

$$(5.72)\qquad V_r(r,\bar{\alpha}) > V_r(r,\alpha_i),$$

that is , in this example, $\bar{\alpha} > \alpha_i$. Hence, for the r(t)-process starting at the initial value $r_0 < \sqrt{(8-3\sigma^2)/6\bar{\alpha}}$ and $\alpha_i < \bar{\alpha}$, it may be concluded that

$$(5.73)\qquad P_r\{\ \lim_{t\to\infty} r(t)=0 \mid r(0)=r_0,\alpha(0)=\alpha_i\ \}$$
$$\geq 1 - W_\alpha(r_0)\Big/W_\alpha(\sqrt{\frac{8-3\sigma^2}{6\bar{\alpha}}}).$$

5.7 Summary

In this chapter, a new approach has been developed to analyze the asymptotic stability of nonlinear dynamical systems with a random parameter behaving as a white Gaussian process. The basic notion presented here was a choice of the stochastic Lyapunov function with an advantage that influences of initial values of the system states came out.

Introducing the concept of random evolution associated with the limit theorem, the stability analysis was extended to a general class of nonlinear dynamical systems involving two kinds of random parameter modeled by a white Gaussian and a Markov chain processes respectively.

Throughout this chapter, the relation between the asymptotic behavior of nonlinear stochastic systems and the domain of their initial values was examined by using the useful Theorems giving sufficient conditions for the asymptotic stability with the probability appraisal.

Chapter 6

Noise Stabilization of Nonlinear Dynamical Systems

6.1 Introduction

The idea of noise stabilization originates from the interesting fact that the inverted pendulum can be stabilized whose base is subjected to a periodic vertical displacement with a zero mean. That is, as shown in Fig.6.1, we shall consider a simple pendulum of length ℓ, mass m and damping coefficient c, and let q be the

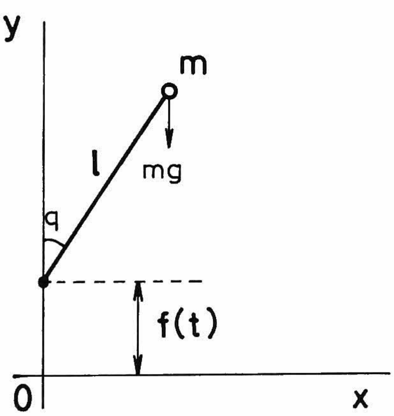


Fig.6.1 Schematic Representation of an Inverted Pendulum

angular displacement from upward vertical. Then, the equation of motion of the pendulum becomes

$$(6.1) \quad \ddot{q} + (2c/m)\dot{q} - \ell^{-1}(\ddot{f} + g)\sin q = 0$$

where g is the acceleration of gravity and f(t) is the imposed base displacement.

When the base motion f(t) is sinusoidal, the stability of the inverted pendulum is determined from the Mathieu equation [84]. Hemp and Sethna[84] obtained additional results when the base motion f(t) is almost periodic and periodic forcing terms appear in the right-hand side. As an extension, it is the problem whether or not the pendulum can be stabilized with a base motion that f(t) is a sample function from some type of stochastic process. Bogdanoff and Citron[43] derived conditions for stability when the base motion f(t) was a second-order stationary, random parametric excitation having a discrete-power spectral density and demonstrated their results with a physical experiment. Mitchell[48] derived sufficient conditions for the sample stability of the linear inverted pendulum equation together with a base motion that is a sample function from a stochastic process with a continuous power spectral density function.

However, these studies were restricted to the stability of the inverted pendulum and a noise stabilizing signal was given as the forcing function. From the fact that there are, in practice, many inherently unstable nonlinear control or dynamical systems and these have to operate in random environment, in this chapter, we shall explore the possibility of noise stabilization for a more general class of nonlinear dynamical systems. Thus, if sufficient-

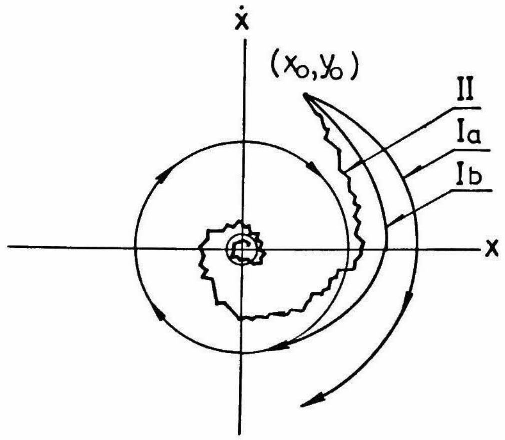


Fig.6.2 Illustration of Noise Stabilization

ly comprehensive conditions can be ascertained for which an unstable state can be made stable, then it is conceivable that the introduction of noise generated by various components in a system may be used for its own stabilization. Hence, a class of self-stabilized system may be possible.

6.2 Basic Equation and Problem Statement

We shall consider the second-order nonlinear stochastic differential equation,

$$(6.2) \qquad \ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = -\delta h(x,\dot{x})\dot{f}(t).$$

Equation (6.2) may be considered as a generalization of mathematical model of dynamical systems. The simplest example is an inverted pendulum of Eq.(6.1) where $\omega^2 x + \varepsilon g(x,\dot{x}) = (2c/m)\dot{x} - (g/\ell)\sin x$ and $h(x,\dot{x}) = -\sin x$. Taking into account practical examples, in Eq.(6.2), the coefficients β, ε, δ might be constants,

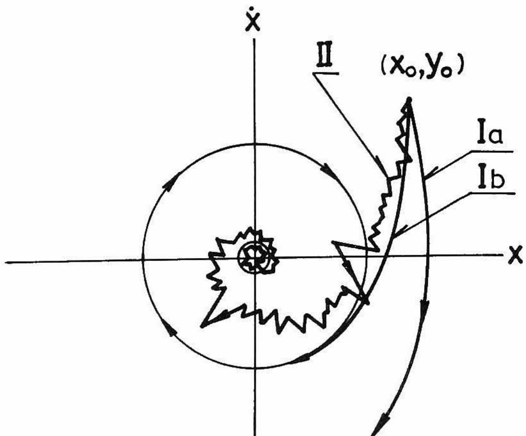


Fig.6.2 Illustration of Noise Stabilization

$g(x,\dot{x})$ and $h(x,\dot{x})$ are nonlinear functions respectively and $\dot{f}(t)$ a white Gaussian noise process.

Now, suppose that the deterministic system with $\dot{f}(t)=0$,

(6.3) $\quad \ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = 0$

is unstable. As shown in curves I_a and I_b of Fig.6.2, the solution process of Eq.(6.3) diverges or shows a limit cycle behavior. Then, the problem is, as shown in the curve II of Fig.6.2, to stabilize the system described by Eq.(6.3) through the addition of a random noise term $-\delta h(x,\dot{x})\dot{f}(t)$. Hence, the form of the function $h(x,\dot{x})$ must be found out.

Let state variables be $x=x_1$ and $\dot{x}_1=x_2$ respectively. Equation (6.2) is expressed by the nonlinear stochastic differential equation of Itô-type,[66]

(6.4a) $\quad dx_1 = x_2 dt$

(6.4b) $\quad dx_2 = -\{\omega^2 x_1 + \varepsilon g(x_1,x_2)\}dt - \delta h(x_1,x_2)dw(t),$

where the w(t)-process is the Brownian motion process with the following properties; $E[dw(t)]=0$, $E[\{dw(t)\}^2]=\sigma^2 dt$.

Equation (6.4) is the basic equation of the present study. It is apparent that the two-dimensional solution process of Eq. (6.4), $x=(x_1,x_2)'$, is a uniform Markov process. Let $p_\varepsilon(x_1,x_2;\tau;x_1^0,x_2^0)$ be the probability density of a transition from the point (x_1,x_2) to the point (x_1^0,x_2^0) in time τ for the trajectory of the x(t)-process. It is also well-known that the probability density p_ε satisfies

(6.5) $$-\frac{\partial p_\varepsilon}{\partial \tau} = x_2\frac{\partial p_\varepsilon}{\partial x_1} + \{-\omega^2 x_1 - \varepsilon g(x_1,x_2)\}\frac{\partial p_\varepsilon}{\partial x_2} + \frac{\delta^2\sigma^2}{2}h^2(x_1,x_2)\frac{\partial^2 p_\varepsilon}{\partial x_2^2}.$$

Letting $\delta^2=\varepsilon$ and rearranging Eq.(6.5), we have

$$(6.6)\quad -\frac{\partial p_\varepsilon}{\partial \tau} = x_2\frac{\partial p_\varepsilon}{\partial x_1} - \omega^2 x_1\frac{\partial p_\varepsilon}{\partial x_2} + \varepsilon[-g(x_1,x_2)\frac{\partial p_\varepsilon}{\partial x_2} + \frac{\sigma^2}{2}h^2(x_1,x_2)\frac{\partial^2 p_\varepsilon}{\partial x_2^2}]$$

with the initial condition

$$(6.7)\quad p_\varepsilon(x_1,x_2;0;x_1^0,x_2^0) = \delta(x_1-x_1^0,x_2-x_2^0),$$

where δ is Dirac's delta function. With the help of basic knowledge in probability theory, it may easily be shown that the density for the stationary distribution of the x(t)-process, $p_\varepsilon(x_1^0,x_2^0)$, is defined by

$$(6.8a)\quad p_\varepsilon^0(x_1^0,x_2^0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_\varepsilon^0(x_1,x_2;\tau;x_1^0,x_2^0)dx_1dx_2$$

where it is obvious that

$$(6.8b)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_\varepsilon^0(x_1^0,x_2^0)dx_1^0dx_2^0 = 1.$$

Assuming the existence of the stationary probability density function*1 in Eq.(6.6), it can be written that

$$(6.9)\quad -\varepsilon[-g(x_1,x_2)\frac{\partial p_\varepsilon^0}{\partial x_2} + \frac{\sigma^2}{2}h^2(x_1,x_2)\frac{\partial^2 p_\varepsilon^0}{\partial x_2^2}] - x_2\frac{\partial p_\varepsilon^0}{\partial x_1} + \omega^2 x_1\frac{\partial p_\varepsilon^0}{\partial x_2} = 0.$$

Introducing the polar coordinate (r,ψ) instead of the rectangular coordinate (x_1,x_2) along the relation,

$$(6.10)\quad x_1 = \frac{r}{\omega}\sin(\psi-\omega t),\quad x_2 = -r\cos(\psi-\omega t),$$

---

*1 This assumption is very important for realizing the noise stabilization. The existence of the stationary probability density can be examined based on the existence theorems of stationary responses described in Chapter 3.

Eq.(6.6) becomes

$$(6.11)\quad \frac{\partial u_\varepsilon}{\partial \tau} = -\varepsilon[g\{\frac{r}{\omega}\sin(\psi-\omega t),-r\cos(\psi-\omega t)\}\{-\cos(\psi-\omega t)\frac{\partial u_\varepsilon}{\partial r}$$

$$+ \frac{\sin(\psi-\omega t)}{r}\frac{\partial u_\varepsilon}{\partial \psi}\} + \frac{\sigma^2}{2}h^2\{\frac{r}{\omega}\sin(\psi-\omega t),-r\cos(\psi-\omega t)\}$$

$$\times\{\cos^2(\psi-\omega t)\frac{\partial^2 u_\varepsilon}{\partial r^2} - \frac{\sin2(\psi-\omega t)}{r}\frac{\partial^2 u_\varepsilon}{\partial r\partial\psi} + \frac{\sin^2(\psi-\omega t)}{r^2}\frac{\partial^2 u_\varepsilon}{\partial \psi^2}$$

$$+ \frac{\sin2(\psi-\omega t)}{r^2}\frac{\partial u_\varepsilon}{\partial \psi} + \frac{\sin^2(\psi-\omega t)}{r}\frac{\partial u_\varepsilon}{\partial r}\}]$$

where u_ε denotes the porbability density function of a transition with respect to the new coordinate (r,ψ), i.e.,

$$(6.12)\quad u_\varepsilon(r_1,\psi_1;\tau;r_0,\psi_0) \equiv p_\varepsilon\{(\frac{r_1}{\omega}\sin(\psi_1-\omega t),-r\cos(\psi_1-\omega t);\tau;$$

$$\frac{r_0}{\omega}\sin(\psi_0-\omega t'),-r_0\cos(\psi_0-\omega t')\}$$

where $t+\tau=t'$.

6.3 Application of Averaging Principle

Let $p_0(r,\psi;\tau;r_0,\psi_0)$ be the probability density of a transition from the point (r,ψ) to the point (r_0,ψ_0) in time τ for the trajectory of the (r,ψ)-process. The application of the averaging principle in Section 2.2 to Eq.(6.11) brings us the result,

$$(6.13)\quad \frac{\partial p_0}{\partial \tau} = -\varepsilon[\frac{\sigma^2}{2}\{A(r)\frac{\partial^2 p_0}{\partial r^2} + \frac{1}{r}B(r)\frac{\partial p_0}{\partial r} + \frac{1}{r^2}B(r)\frac{\partial^2 p_0}{\partial \psi^2} + \frac{1}{r^2}C(r)\frac{\partial p_0}{\partial \psi}$$

$$- \frac{1}{r}C(r)\frac{\partial^2 p_0}{\partial r\partial\psi}\} + \Phi(r)\frac{\partial p_0}{\partial r} + \frac{1}{r}\Psi(r)\frac{\partial p_0}{\partial \psi}]$$

where, with the symbol $\psi-\omega t=\theta+\pi/2$.

(6.14a) $$A(r) = \frac{1}{2\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\sin^2\theta d\theta$$

(6.14b) $$B(r) = \frac{1}{2\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\cos^2\theta d\theta$$

(6.14c) $$C(r) = \frac{1}{2\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\sin 2\theta d\theta$$

(6.14d) $$\Phi(r) = \frac{1}{2\pi}\int_0^{2\pi} g(\frac{r}{\omega}\cos\theta, r\sin\theta)\sin\theta d\theta$$

(6.14e) $$\Psi(r) = \frac{1}{2\pi}\int_0^{2\pi} g(\frac{r}{\omega}\cos\theta, r\sin\theta)\cos\theta d\theta.$$

According to the relation (6.12), it may be concluded that

(6.15) $$\lim_{\varepsilon\to 0} \sup_{t\in I_T} |u_\varepsilon(r,\psi;\tau;r_0,\psi_0) - p_0(r,\psi;\tau;r_0,\psi_0)| = 0.$$

For the process whose probability density of a transition is determined by Eq.(6.13), if there exists a stationary probability density $p(r,\psi)$, then it is evidently independent of ψ. Thus the stationary probability density is a solution of the differential equation,

(6.16a) $$U^2(r)\frac{d^2p}{dr^2} + V(r)\frac{dp}{dr} = 0$$

with the condition,

(6.16b) $$\int_0^\infty p(r)rdr = 1$$

where

(6.17) $$U^2(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\sin^2\theta d\theta$$

$$(6.18) \quad V(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\frac{\cos^2\theta}{r}d\theta$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} g(\frac{r}{\omega}\cos\theta, r\sin\theta)\sin\theta d\theta$$

and where $r \geq 0$.

From Eq.(6.16a), the r(t)-process is obviously Markov process with the differential generator,

$$(6.19) \quad L_r = U^2(r)\frac{d^2}{dr^2} + V(r)\frac{d}{dr}$$

where both the drift coefficient V(r) and the diffusion coefficient $U^2(r)$ are polynomials with respect to r and satisfy Lipschitz and the uniform growth conditions. Thus the sample process of the r(t)-process exists, is unique and is continuous with probability one.

6.4 Stability Criteria associated with Singularities

With the help of knowledges described in Section 2.1 of Chapter 2, new theorems of stability are established based on the classification of singularities of the one-dimensional diffusion process with the differential generator (6.19). Before the statement of Theorem 6.1, we need the following Lemma 6.1 which is a slight extension of the lemma by Itô and McKean[85].

[Lemma 6.1] Consider the regular interval $I=[r_1,r_2]$ such that $s(r_1)$ and $s(r_2)$ are finite. Let $P_r(a;r_1,r_2)$ denote the probability that the process r(t) originating at a fixed $a \in I$ reaches the point r_1 before reaching the point r_2. Then the canonical scale s(a) in I is uniquely determined by

$$(6.20) \qquad P_r(a;r_1,r_2) = \frac{s(a) - s(r_2)}{s(r_1) - s(r_2)} .$$

The proof is omitted because this lemma is only an extension of the lemma by Itô and McKean[85].

[Theorem 6.1] Let be the r(t)-process with the singular point $r=r_1$ satisfying $U^2(r)=0$ in (6.19). The singular point $r=r_1$ on the interval $I=[r_1,r_2]$ where $r_1<r_2$ is stable in probability, if the following two conditions are satisfied:

(1) $r=r_1$ is 'exit boundary' ($r_1 \neq 0$)

(2) $r=r_2$ is 'natural boundary' and $s(r_2)=+\infty$, or 'entrance boundary'.

Thus, (6.19) has a stable singular point.

(Proof) We shall first show the sufficiency. Letting $a=r_0 (r_1 \leq r_0 \leq r_2)$, it follows from Lemma 6.1 and the condition (1) that

$$(6.21) \qquad \lim_{r_0 \to r_1} P_r(r_0;r_1,r_2) = 1.$$

The equality (6.21) implies that

$$(6.22) \qquad \lim_{r_0 \to r_1} P_r\{ \sup_{t \geq 0} |r(t;r_0) - r_1| > \varepsilon \} = 0.$$

Thus, the point $r=r_1$ is a trap. From the properties of diffusion processes described in Definitions 2.1,2.2,2.3 and 2.4 of Section 2.1 in Chapter 2, the probability that sample paths of the r(t)-process stay on the interval $[r_1,r_2]$ is zero. Similarly, by the condition (2), the sample process r(t) can not reach $r=r_2$. Therefore, for the singular point $r=r_1$ to be stable in probability, it is sufficient that the conditions (1) and (2) are satisfied.

Next, we shall prove the necessity. For the singular point $r=r_1$ to be stable in probability, it must hold that

$$(6.23) \qquad P_r\{ \lim_{t\to\infty} |r(t;r_0) - r_1| = 0 \} = 1.$$

It is obvious that the equality (6.23) holds if and only if the point $r=r_1$ is the exit boundary and the point $r=r_2$ is the locally unattractive natural boundary or entrance boundary.

Then, Eq.(6.19) has a stable singular point.

[Theorem 6.2] Suppose that Theorem 6.1 holds. If the point $r=r_1$ approaches to the neighborhood of the origin and $r_2\to\infty$, then the system (6.4) is asymptotically stable in probability.

(Proof) By Theorem 6.1, the conditions (1) and (2) are given. Since the point $r=r_1$ is the exit boundary and the point $r=r_2$ is the locally unattractive natural boundary or entrance boundary, the sample process $r(t)$ reaches the point $r=r_1$ within a finite time and after that, they stay forever at $r=r_1$ with probability one. This situation implies that

$$(6.24) \qquad \lim_{\tau\to\infty} P_r\{ \sup_{t\geq\tau} |r(t;r_0) - r_1| > \varepsilon \} = 0.$$

Furthermore, if the point $r=r_1$ approaches to the neighborhood of the origin $r=0$, then, from the equality (6.24), we have

$$(6.25) \qquad \lim_{\tau\to\infty} P_r\{ \sup_{t\geq\tau} r(t;r_0) > \varepsilon \} = 0.$$

From Theorem 6.1 and the equality (6.25), it is obvious that the r(t)-process is asymptotically stable in probability. This shows that the system (6.4) is asymptotically stable in probability.

6.5 Selection of Stabilizing Signals

Let $g(x,\dot{x})$ in Eq.(6.2) be given by

$$(6.26) \qquad g(x,\dot{x}) \equiv g(x_1,x_2) = \alpha x_1 + \rho x_1^3$$

where α and ρ are constant values. Then, it is well-known that the system,

$$(6.27) \qquad \ddot{x} + \omega^2 x + \varepsilon(\alpha x + \rho x^3) = 0$$

is a mathematical model of Duffing-type nonlinear system which has a stable limit cycle. Also, the response of the system,

$$(6.28) \qquad \ddot{x} + \omega^2 x + \varepsilon(\alpha x + \rho x^3) = -\delta\dot{\xi}(t)$$

shows unstable behavior such that the noise term $\dot{\xi}(t)$ adds to a stable limit cycle of Eq.(6.27). Based on the fact that the systems described by Eqs.(6.27) and (6.28) have unstable characteristics, we shall look for a stabilizing signal $\delta h(x,\dot{x})\dot{\xi}(t)$ in such a way that the unstable systems (6.27) and (6.28) become stable. A few trials which will be mentioned in this section lead the reader to interesting results in which the unstable systems become stable.

6.5.A Biased Sinusoidal Signal

Let the stabilizing term $h(x_1,x_2)$ be chosen by

$$(6.29) \qquad h(x_1,x_2) = a\left[\left|\cos b\sqrt{\omega^2 x_1^2+x_2^2}\Big/\sqrt{\omega^2 x_1^2+x_2^2} - k\right|\right]^{1/2},$$

or, in the polar coordinate (r,ψ), by using Eq.(6.10), (6.30) $$h(r) = a\sqrt{\left|\frac{\cos br}{r} - k\right|}\ ,$$

where a,b and k are constants. Then, with the help of (6.26) and (6.30), Eq.(6.4) becomes

(6.31a) $$dx_1 = x_2 dt$$

(6.31b) $$dx_2 = -\{\omega^2 x_1 + \varepsilon(\alpha x_1 + \rho x_1^3)\}dt + \delta a\sqrt{\left|\frac{\cos br}{r} - k\right|}dw\ .$$

From Eqs.(6.17) and (6.18), the diffusion and the drift coefficients respectively have the form,

(6.32) $$U^2(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} a^2\left|\frac{\cos br}{r} - k\right|\sin^2\theta d\theta$$

$$= \frac{\sigma^2 a^2}{4}\left|\frac{\cos br}{r} - k\right|,$$

(6.33) $$V(r) = \frac{\sigma^2}{2}\frac{1}{2\pi}\int_0^{2\pi} a^2\left|\frac{\cos br}{r} - k\right|\frac{\cos^2\theta}{r}d\theta + \Phi(r)$$

$$= \frac{\sigma^2 a^2}{4r}\left|\frac{\cos br}{r} - k\right|.$$

Our first task is to find the singular point $r=r_s$ satisfying $U^2(r)=0$, i.e.,

(6.34) $$\left|\frac{\cos br}{r} - k\right| = 0.$$

From Eq.(6.34), it is obvious that there are a lot of singular points along the line of r. In order to realize noise stabilization, it is necessary to obtain only one stable singular point at or near the origin. This implies that the spectral density of white Gaussian noise may be transformed to the spectral density with the finite band-width to stabilize the system. To do this, we select a parameter k by the graphical method as shown in Fig.6.3. According to Fig.6.3, it may numerically be observed that, for $k \geq 0.16b$, there exists a singular point. However, it is very difficult to proceed the calculations of Eqs.(2.20a) to (2.20d) at this singular point by applying Eq.(6.30) directly. Then, we shall demonstrate an approximation method for the calculations of Eqs.(2.20a) to (2.20d). For this purpose, the relation between $4U^2(r)/\sigma^2 a^2$ and ζ (=br) is examined by using Eq.(6.32). The result is shown in Fig.6.4 where the point P is a singular point. Here, the value of k was set as k=0.2 from the result of Fig.6.3. Observing that the singular point P is closely located at the origin in Fig.6.4, the following approximation can be performed by the least square method,

$$(6.35) \qquad \frac{\cos br}{r} \cong \frac{b}{6}\{(\zeta - 3)^2 - 2\}.$$

This approximation allows us to evaluate approximately the location of singular points. Substituting the approximation (6.35) into Eq.(6.32), it follows that

$$(6.36) \qquad U^2(r) \cong \frac{\sigma^2 a^2}{4}\left|\frac{b}{6}\{(\zeta - 3)^2 - 2\} - k\right|$$

From Eqs.(2.14) and (6.36), the singular point is

$$(6.37) \qquad r_s = \frac{1}{b}\left(3 - \sqrt{2 + \frac{6k}{b}}\right).$$

In order to classify the singular point, the following two cases should be considered:

[1] Interval $[r_s, \infty)$

Substituting (6.35) into Eqs.(6.32) and (6.33), we have,

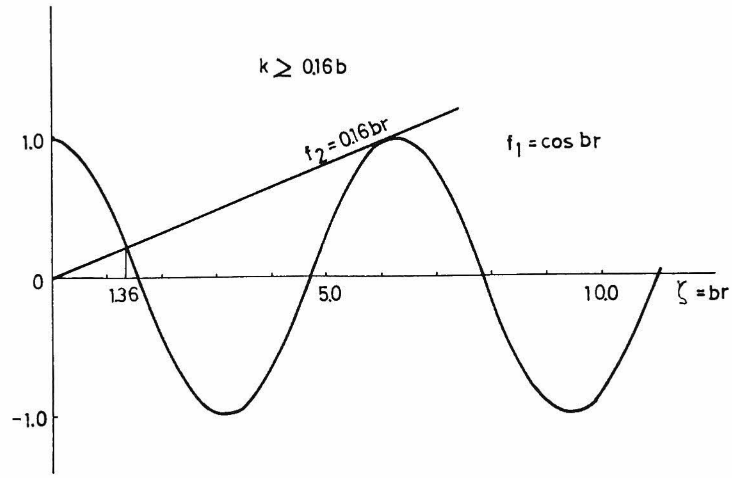


Fig.6.3 Determination of a parameter k to realize one singular point.

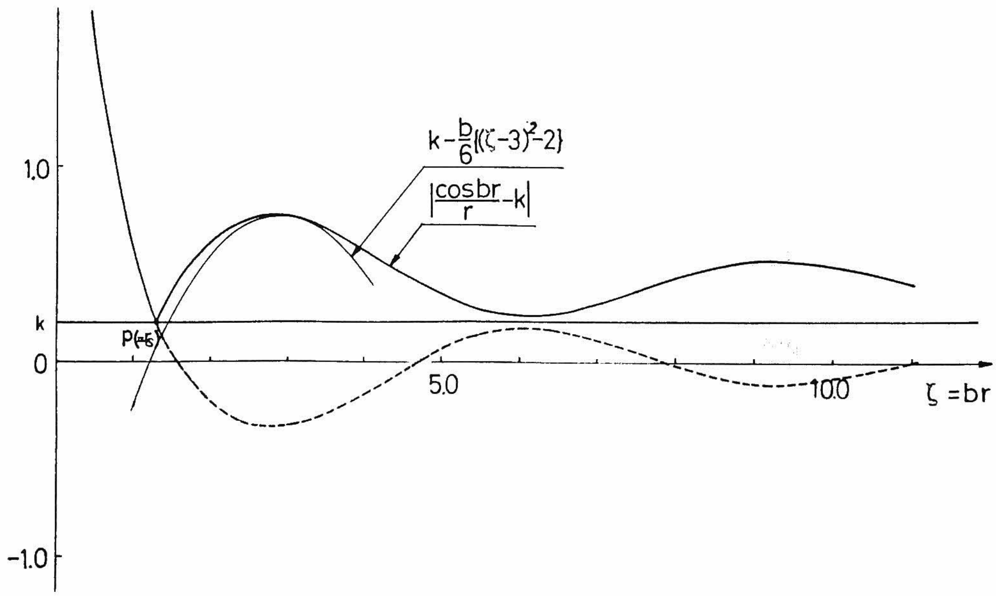


Fig.6.4 Approximation to cosbr/r in the neighborhood of a singular point $P(=r_s)$.

(6.38) $$U^2(r) \cong \frac{\sigma^2 a^2}{4}[-\frac{b}{6}\{(\zeta - 3)^2 - 2\} + k]$$

and

(6.39) $$V(r) \cong \frac{\sigma^2 a^2}{4r}[-\frac{b}{6}\{(\zeta - 3)^2 - 2\} + k].$$

Application of Eqs.(6.38) and (6.39) to Eq.(2.17) gives

(6.40) $$B_s(r) = \int_{br_0}^{br} V(\zeta)U^{-2}(\zeta)d\zeta = \ell og\frac{r}{r_0} .$$

From Eqs.(2.18),(2.19),(6.38) and (6.40), the canonical scale and the canonical measure are respectively

(6.41) $$ds(r) = \frac{r_0}{\zeta} d\zeta$$

and

(6.42) $$dm(r) = \frac{24}{\sigma^2 a^2 b^3 r_0} \frac{\zeta}{\{-(\zeta-3)^2+k_1\}} d\zeta$$

where

(6.43) $$k_1 = 2 + 6k/b.$$

From Eqs.(6.41) and (6.42), the calculations of Eqs.(2.20a) and (2.20b) give the result,

(6.44) $$\sigma_1 = \int\int_{br_s<\zeta_z<\zeta_y<br_1'} dm(\zeta_y)ds(\zeta_z)$$

$$= \frac{12}{\sigma^2 a^2 b^3}(1-\frac{3}{\sqrt{k_1}})[\ell og\{\sqrt{k_1}+(\zeta_z-3)\}\ell og|\frac{\zeta_z}{\sqrt{k_1}-3}|+\frac{\sqrt{k_1}+(\zeta_z-3)}{\sqrt{k_1}-3}$$

$$+\frac{\{\sqrt{k_1}+(\zeta_z-3)\}^2}{4(\sqrt{k_1}-3)^2}+\frac{\{\sqrt{k_1}+(\zeta_z-3)\}^3}{9(\sqrt{k_1}-3)^3}+\cdots]_{br_s}^{br_1'}+\frac{12}{\sigma^2 a^2 b^3}(1+\frac{3}{\sqrt{k_1}})$$

$$\times[\ell og\{\sqrt{k_1}-(\zeta_z-3)\}\ell og|\frac{\zeta_z}{\sqrt{k_1}+3}| + \frac{\sqrt{k_1}-(\zeta_z-3)}{\sqrt{k_1}+3} + \frac{\{\sqrt{k_1}-(\zeta_z-3)\}^2}{4(\sqrt{k_1}+3)^2}$$

$$+ \frac{\{\sqrt{k_1}-(\zeta_z-3)\}^3}{9(\sqrt{k_1}+3)^3} + \cdots]_{br_s}^{br_1'} + \frac{12}{\sigma^2 a^2 b^3}(A_1+3A_2)\ell og\frac{r_s}{r_1'} < \infty$$

where

$$(6.45) \quad A_1 = \ell og\{k_1 - (br_1' - 3)^2\},$$

$$(6.46) \quad A_2 = \frac{1}{\sqrt{k_1}}\ell og\frac{\sqrt{k_1}+(br_1'-3)}{\sqrt{k_1}-(br_1'-3)} ,$$

and

$$(6.47) \quad \mu_1 = \iint_{br_s<\zeta_z<\zeta_y<br_1'} ds(\zeta_y)dm(\zeta_z)$$

$$= \frac{24}{\sigma^2 a^2 b^3}\{\frac{1}{2}(1+\frac{3}{\sqrt{k_1}})[-\ell og\zeta_z \ell og|\frac{\sqrt{k_1}-(\zeta_z-3)}{\sqrt{k_1}+3}| - \frac{\zeta_z}{\sqrt{k_1}+3}$$

$$- \frac{\zeta_z^2}{4(\sqrt{k_1}+3)^2} - \frac{\zeta_z^3}{9(\sqrt{k_1}+3)^3} - \cdots]_{br_s}^{br_1'} - \frac{1}{2}(1-\frac{3}{\sqrt{k_1}})[\ell og\zeta_z$$

$$\times\ell og|\frac{\sqrt{k_1}+(\zeta_z-3)}{\sqrt{k_1}-3}| + \frac{\zeta_z}{\sqrt{k_1}-3} + \frac{\zeta_z^2}{4(\sqrt{k_1}-3)^2} + \frac{\zeta_z^3}{9(\sqrt{k_1}-3)^3} + \cdots]_{br_s}^{br_1'}$$

$$+ [- \frac{1}{2}(1-\frac{3}{\sqrt{k_1}})\ell og|\sqrt{k_1}-(\zeta_z-3)|\ell og br_1' - \frac{1}{2}(1+\frac{3}{\sqrt{k_1}})$$

$$\times\ell og|\sqrt{k_1}+(\zeta_z-3)|]_{br_s}^{br_1'}\} = \infty.$$

From the results of Eqs.(6.44) and (6.47), it may be found that the singular point $r=r_s$ is the exit boundary.

Furthermore, we shall consider the property of the boundary

$r=\infty$. In this case, since $\cos br/r$ converges to zero as $r\to\infty$, $U^2(r)$ and $V(r)$ are respectively approximated by

$$(6.48)\qquad U^2(r) \cong \frac{\sigma^2 a^2}{4} k$$

and

$$(6.49)\qquad V(r) \cong \frac{\sigma^2 a^2}{4r} k \; .$$

Therefore, the calculation of Eq.(2.17) gives,

$$(6.50)\qquad B_s(r) = \int_{r_0}^{r} \frac{1}{r'} dr' = \ell og \frac{r}{r_0} \; .$$

Using the approximations (6.48) and (6.49), we have

$$(6.51)\qquad \sigma_2 = \iint_{r_2'<y<z<\infty} ds(y)dm(z) = \infty$$

$$(6.52)\qquad \mu_2 = \iint_{r_2'<y<z<\infty} dm(y)ds(z) = \infty,$$

and it is easily obtained that

$$(6.53)\qquad s(r=\infty) = \infty.$$

Then, based on Feller's classification criteria of boundaries in Section 2.1, it may be concluded that the point $r=\infty$ is a locally unattractive natural boundary.

[2] Interval $[\,0\,,\,r_s\,]$

In the case when $r=r_s$, it is obvious, from the discussion mentioned above, that the singular point $r=r_s$ is the exit boundary. We shall now examine the property of the boundary $r=0$. In the neighborhood of $r=0$, as shown in Fig.6.4, the approximation $h^2(r) \cong a^2 k_0/r$ (where k_0 is a constant) is introduced. From this approximation, we have

$$(6.54)\qquad U^2(r) = \frac{\sigma^2 a^2}{4}\frac{k_0}{r}$$

$$(6.55)\qquad V(r) = \frac{\sigma^2 a^2}{4}\frac{k_0}{r^2}.$$

Hence, from (6.54) and (6.55), we have

$$(6.56)\qquad ds(r) = \frac{r_0}{r}$$

$$(6.57)\qquad dm(r) = \frac{4r^2}{\sigma^2 a^2 k_0 r_0}.$$

Therefore, when r=0, it is a simple exercise to obtain

$$(6.58)\qquad \sigma_1(0) = \infty \quad \text{and} \quad \mu_1(0) < \infty.$$

Then, it follows that the origin r=0 is the entrance boundary.

The results obtained in 6.5.A are summarized in Table 6.1. Also, the illustration of sample behaviors based on Table 6.1 is shown in Fig.6.5.

Table 6.1 Classification of boundaries for the r(t)-process determined by Eq.(6.31).

σ_i	μ_i	Classification of boundaries	
$\sigma_1 = \infty$	$\mu_1 < \infty$	"entrance" at $r=0$	for $[0,r_s]$
$\sigma_2 < \infty$	$\mu_2 = \infty$	"exit" at $r = r_s$	
$\sigma_1 < \infty$	$\mu_1 = \infty$	"exit" at $r = r_s$	for $[r_s,\infty)$
$\sigma_2 = \infty$	$\mu_2 = \infty$	"natural" (locally unattractive) at $r = \infty$	

Then, with Theorem 6.1 and Theorem 6.2, the process which originates at a point in the interval $[0,\infty)$ arrives at the singular point r_s in a finite time with probability one and stays on r_s forever.

From Eq.(6.37), as the value of b increases, the location of the singular point comes near the origin. This implies that an application of the stabilizing signal with sufficiently high frequency results a smaller amplitude of the limit cycle.

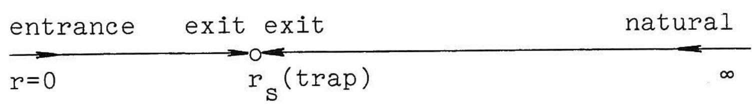


Fig.6.5 Illustration of Sample Behaviors of Eq.(6.31) based on Table 6.1.

6.5.B Biased Polynomial Type Signal

As another type of stabilizing signal, let $h(x_1,x_2)$ be chosen by

$$(6.59)\qquad h(x_1,x_2) = h(r) = a\sqrt{r\,|\,r-c\,|}$$

where a and c are constants. Then, with the help of Eq.(6.26), Eq.(6.4) becomes

$$(6.60a)\qquad dx_1 = x_2dt$$

$$(6.60b)\qquad dx_2 = -\{\omega^2x_1 + \varepsilon(\alpha x_1 + \rho x_1^3)\}dt + \delta a\sqrt{r\,|r-c|}dw.$$

From Eqs.(6.17) and (6.18), the diffusion and the drift coefficients are respectively,

(i) Interval $[0,c]$

(6.61) $\quad U^2(r) = (\sigma^2a^2/4)r(c-r)$

(6.62) $\quad V(r) = (\sigma^2a^2/4)(c-r)$

(ii) Interval $[c,\infty)$

(6.63) $\quad U^2(r) = (\sigma^2a^2/4)r(r-c)$

(6.64) $\quad V(r) = (\sigma^2a^2/4)(r-c)$.

Consequently, from Eqs.(6.61) and (6.63), the singular points are found out to be

(6.65) $\quad r_{s1} = 0$ and $r_{s2} = c$.

Then, from Eq.(2.17), we obtain

(6.66) $\quad B_s(r) = \int_{r_0}^{r} \frac{1}{r'}\,dr' = \log\frac{r}{r_0}$.

[1] Interval $[c,\infty)$

Using Eqs.(6.63) and (6.66), the canonical scale and canonical measure are respectively,

(6.67) $\quad ds(r) = (r_0/r)dr$

and

(6.68) $\quad dm(r) = \frac{4}{\sigma^2a^2r_0}\frac{1}{r-c}\,dr$.

Substituting (6.67) and (6.68) into (2.20a) and (2.20b), it follows that

(6.69) $\quad \sigma_1 = \iint_{r_c<y<x<r_1'} dm(x)ds(y)$

$$= \lim_{r_c \to c} \frac{4}{\sigma^2 a^2} \int_{r_c}^{r_1'} \frac{1}{y} \left(\int_{y}^{r_1'} \frac{1}{x-c} dx \right) dy$$

$$= \lim_{r_c \to c} \frac{4}{\sigma^2 a^2} \{ \ell og(r_1'-c) \ell og \frac{r_1'}{r_c}$$

$$+ [\ell og(y-c) \ell og \frac{y}{c} + \frac{y-c}{c} + \frac{(y-c)^2}{4c^2} + \cdots]_{r_c}^{r_1'} \}$$

$$< \infty$$

and

$$(6.70) \qquad \mu_1 = \iint_{r_c < y < x < r_1'} ds(x) dm(y)$$

$$= \lim_{r_c \to c} \frac{4}{\sigma^2 a^2} \{ \ell og r_1' \ell og \frac{r_1'-c}{r_c-c}$$

$$+ [\ell og y \ell og \frac{y-c}{c} + \frac{y}{c} + \frac{y^2}{4c^2} + \cdots]_{r_c}^{r_1'} \}$$

$$= \infty .$$

Hence, the singular point $r_{s2}=c$ is the exit boundary. We shall classify the case of $r=\infty$. From the results of the calculations of Eqs.(2.20c) and (2.20d), we have

$$(6.71) \qquad \sigma_2 = \iint_{r_2' < x < y < \infty} dm(x) ds(y)$$

$$= \frac{4}{\sigma^2 a^2} \int_{r_2'}^{\infty} \frac{1}{y} \left(\int_{r_2'}^{y} \frac{1}{x-c} dx \right) dy = \infty ,$$

$$(6.72) \qquad \mu_2 = \iint_{r_2' < x < y < \infty} ds(x) dm(y) = \infty$$

and

$$(6.73) \qquad s(r=\infty) = \int_{r_2}^{\infty} \frac{x_0}{x} dx = \infty.$$

Hence, it follows that $r=\infty$ is the natural boundary (locally unattractive).

[2] Interval [0,c]

In the case where the singular point $r_s=c$, it is obvious, from the results of [1], that $r_s=c$ is the exit boundary. We shall classify the property of sample behaviors on r=0. From Eqs.(6.61) and (6.66), we obtain,

$$(6.74) \qquad ds(r) = (r_0/r)dr$$

and

$$(6.75) \qquad dm(r) = \frac{4}{\sigma^2 a^2 r_0} \frac{1}{c-r} dr.$$

Applications of Eqs.(6.74) and (6.75) to Eqs.(2.20a) and (2.20b) give the result;

Table 6.2 Classification of boundaries for the r(t)-process determined by Eq.(6.60)

σ_i	μ_i	Classification of boundaries	
$\sigma_1 = \infty$	$\mu_1 < \infty$	"entrance" at r=0	for [0,c]
$\sigma_2 < \infty$	$\mu_2 = \infty$	"exit" at r=c	
$\sigma_1 < \infty$	$\mu_1 = \infty$	"exit" at r=c	for $[c,\infty)$
$\sigma_2 = \infty$	$\mu_2 = \infty$	"natural" (locally unattractive) at $r=\infty$	

$$(6.76)\qquad \sigma_1 = \iint_{0<y<x<r_1'} dm(x)ds(y)$$

$$= \lim_{r_c \to 0} \frac{4}{\sigma^2 a^2} \int_{r_c}^{r_1'} \frac{1}{y}\left(\int_{y}^{r_1'} \frac{1}{c-x} dx \right) dy = \infty$$

and

$$(6.77)\qquad \mu_1 = \iint_{0<y<x<r_1'} ds(x)dm(y) < \infty.$$

Consequently, the singular point $r_s=0$ is the entrance boundary.

The results obtained in 6.5.B are summarized in Table 6.2. Also, the illustration of sample behaviors based on Table 6.2 is shown in Fig.6.6.

Then, with Theorem 6.1 and Theorem 6.2, the process which starts at a point in the interval $[0,\infty)$ arrives at the singular point $r_s=c$ in a finite time interval with probability one and stays on $r_s=c$ forever.

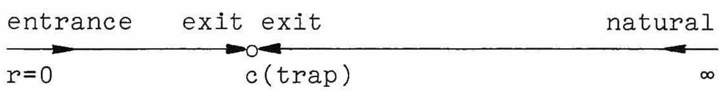


Fig.6.6 Illustration of Sample Behaviors of Eq.(6.60) based on Table 6.2.

6.6 Digital Simulation Studies

Demonstrating the validity of the theoretical viewpoint described above, results of digital simulation studies are shown. First, the sample behavior of the deterministic system will be given, whose process approaches to a stable limit cycle or diverges starting from some initial states. Nextly, introducing the stabilizing noise term at some instant, it may be clarified that the unstable system can be stabilized.

The simulation study was performed in the case of biased sinusoidal stabilizing signal (6.29). We shall suppose that the increment of the state variables of the physical system (6.27) is taken at discrete time t_j and $\delta_j = t_{j+1} - t_j$ $(j=0,1,2,\cdots)$, where δ_j is sufficiently short, i.e.,

(6.78a) $$x_1(j+1) \cong x_2(j+1)\delta_j + x_1(j)$$

(6.78b) $$x_2(j+1) \cong -[\omega^2 x_1(j+1) + \varepsilon\{\alpha x_1(j+1) + \rho x_1(j+1)^3\}]\delta_j + x_2(j).$$

Naturally, the deterministic process $r(j) = \sqrt{x_1(j)^2 + x_2(j)^2}$ determined by Eq.(6.78) is unstable and shows the limit cycle.

Applying the stabilizing signal (6.29), the increments of the state variables are approximately determined by

(6.79a) $$x_1(j+1) \cong x_2(j+1)\delta_j + x_1(j)$$

(6.79b) $$x_2(j+1) \cong -[\omega^2 x_1(j+1) + \varepsilon\{\alpha x_1(j+1) + \rho x_1(j+1)^3\}]\delta_j$$
$$+ \delta a\left|\frac{\cos b\sqrt{\omega^2 x_1(j+1)^2 + x_2(j+1)^2}}{\sqrt{\omega^2 x_1(j+1)^2 + x_2(j+1)^2}} - k\right|^{\frac{1}{2}} \delta w_j + x_2(j)$$

where $\delta w_j = w(j+1) - w(j)$. Recalling the relation $w(t) = \int^t \gamma(s)ds$, δw_j is approximated by $\delta w_j \cong \gamma(j)\delta_j$, where $\gamma(j)$ is the discrete form of white Gaussian noise. We shall use the Gaussian random number $n_1(j)$ with $N[0,1]$, where $n_1(j) = \gamma(j)\delta_j$ [86]. Equation (6.79) can thus be written by

$$(6.80a)\quad x_1(j+1) \cong x_2(j+1)\delta_j + x_1(j)$$

$$(6.80b)\quad x_2(j+1) \cong -[\omega^2 x_1(j+1) + \varepsilon\{\alpha x_1(j+1) + \rho x_1(j+1)^3\}]\delta_j + \delta a\left|\frac{\cos b\sqrt{\omega^2 x_1(j+1)^2+x_2(j+1)^2}}{\sqrt{\omega^2 x_1(j+1)^2+x_2(j+1)^2}} - k\right|^{\frac{1}{2}} n_1(j) + x_2(j).$$

Equation (6.80) was simulated on a digital computer, with a constant step-size δ_j=0.01(sec). A set of parameter values was preassigned as α=-1, ρ=1, ω^2=0.4, ε=0.01 and δ=0.1. The results presented below are representative of the simulation experiments.

A single run of the r(t)-process is shown in Fig.6.7. In Fig.6.7, the dotted line represents unstable behaviors of the system determined by Eq.(6.78) such that the process shows the limit cycle behavior of $r_0 \cong 1.2$, with the initial values of sample processes, $x_1(0)=1.0$ and $x_2(0)=1.0$ or $r(0) \cong 1.2$. At time t=50(sec), the stabilizing signal was applied to the system with the form of the second term of the right-hand side in Eq.(6.80), where a=4, b=2 and k=0.8. The sample path of the stabilized system is shown by the solid curve in Fig.6.7. As stated in 6.5, the r(t)-process arrives at the singular point r_s=0.58. This implies that, by the choice of the value of b, the r(t)-process approaches to the different singular point. From this fact, it can easily be understood that the unstable system was stabilized by applying the stabilizing noise and that the process converges to a limit cycle with a smaller amplitude.

Secondly, a similar simulation study was performed in the case of biased polynomial stabilizing signal (6.59). In this case,

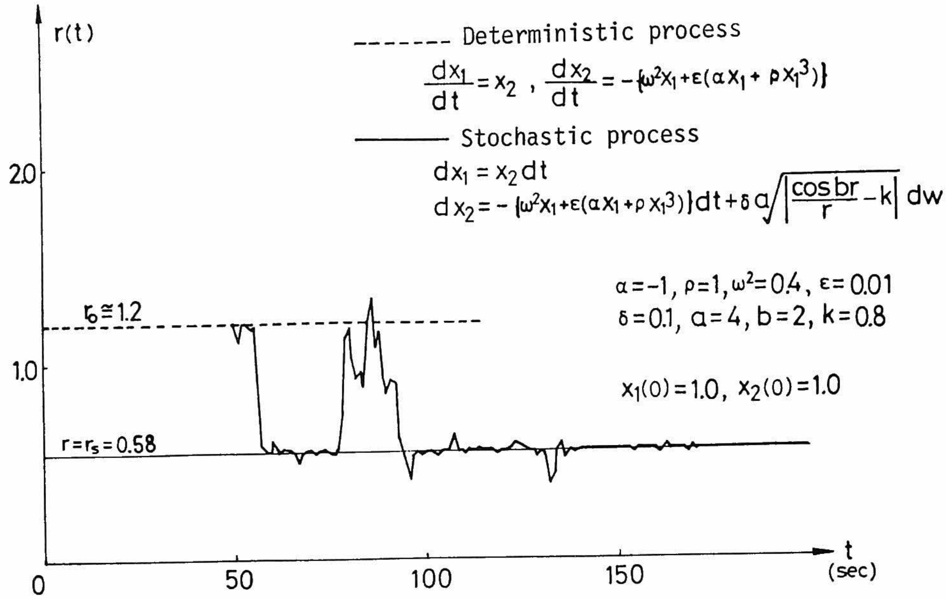
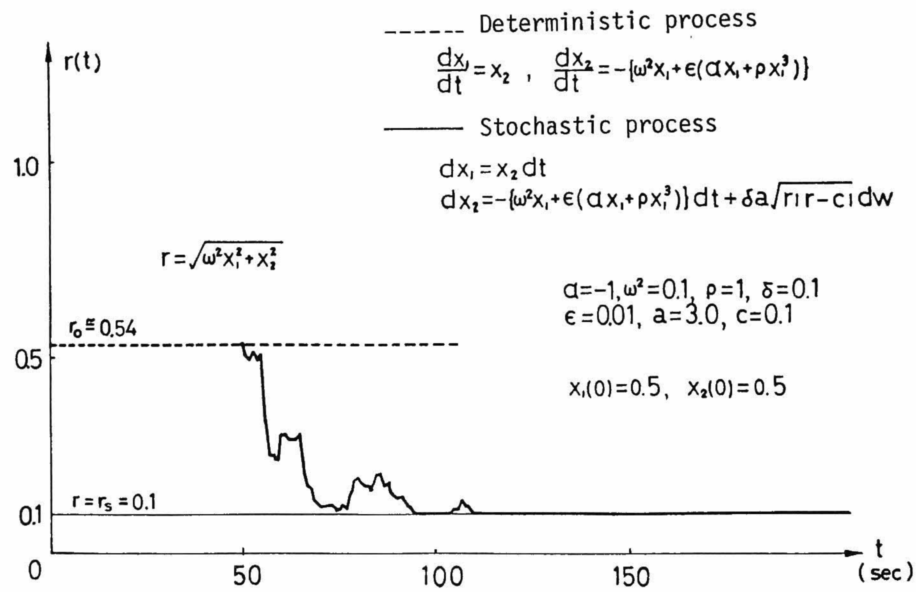
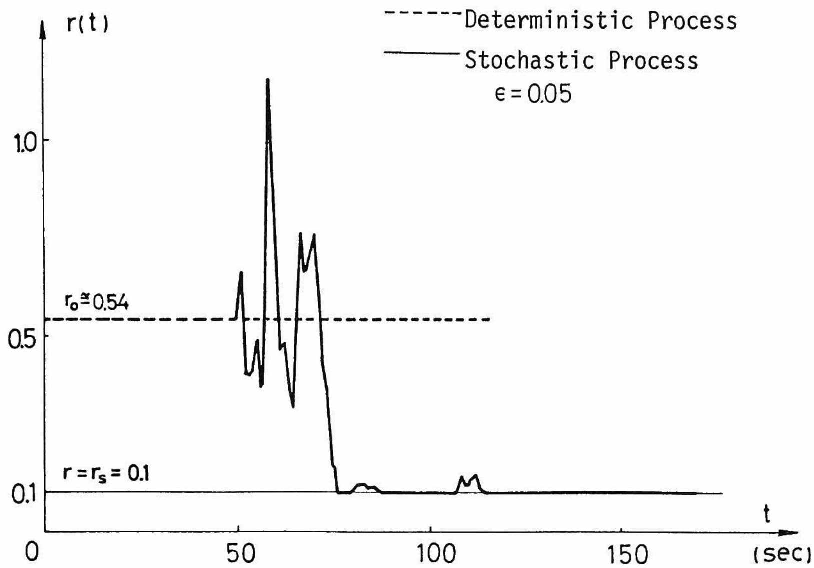


Fig.6.7 A Stabilized Sample Path Behavior of Nonlinear Dynamical System described by Eq.(6.31)

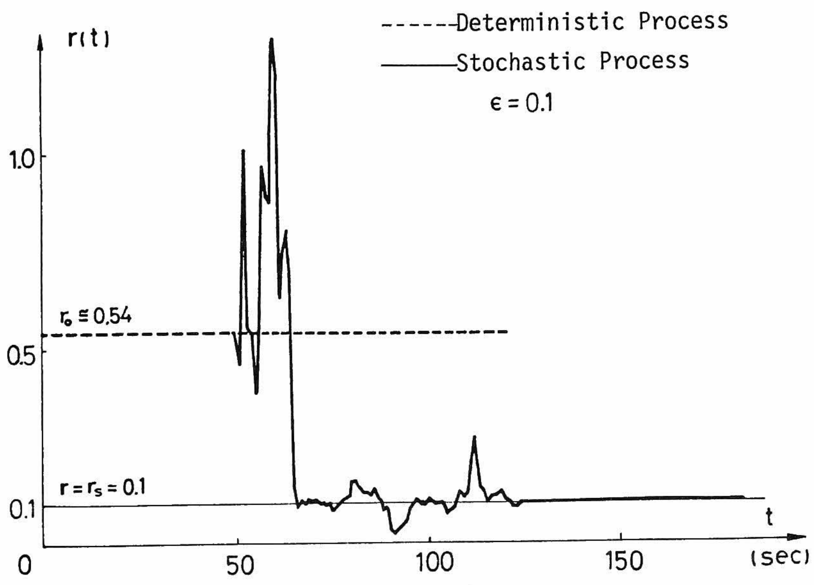


(a) ϵ = 0.01

Fig.6.8 A Stabilized Sample Path Behavior of Nonlinear Dynamical System described by Eq.(6.60)


r(t)
------Deterministic Process
Stochastic Process
ϵ = 0.05
1.0
$r_0 \cong 0.54$
0.5
$r = r_s = 0.1$
0.1
0
50
100
150
t
(sec)

(b) ϵ = 0.05


r(t)
------Deterministic Process
Stochastic Process
ϵ = 0.1
1.0
$r_0 \cong 0.54$
0.5
$r = r_s = 0.1$
0.1
0
50
100
150
t
(sec)

(c) ϵ = 0.1

increments of the state variables are computed by

(6.81a) $x_1(j+1) \cong x_2(j+1)\delta_j + x_1(j)$

(6.81b) $x_2(j+1) \cong -[\omega^2 x_1(j+1) + \varepsilon\{\alpha x_1(j+1) + \rho x_1(j+1)^3\}]\delta_j$

$$+ \delta a[\{\omega^2 x_1(j+1)^2 + x_2(j+1)^2\}^{\frac{1}{2}} | \{\omega^2 x_1(j+1)^2 + x_2(j+1)^2\}^{\frac{1}{2}} - c|]^{\frac{1}{2}}$$

$$\times n_1(j)\delta_j + x_2(j).$$

Sample path behaviors of the r(t)-process is shown in Fig.6.8. Here, the dotted curve with the initial conditions $x_1(0)=0.5$ and $x_2(0)=0.5$ is the unstable run with the limit cycle of the amplitude

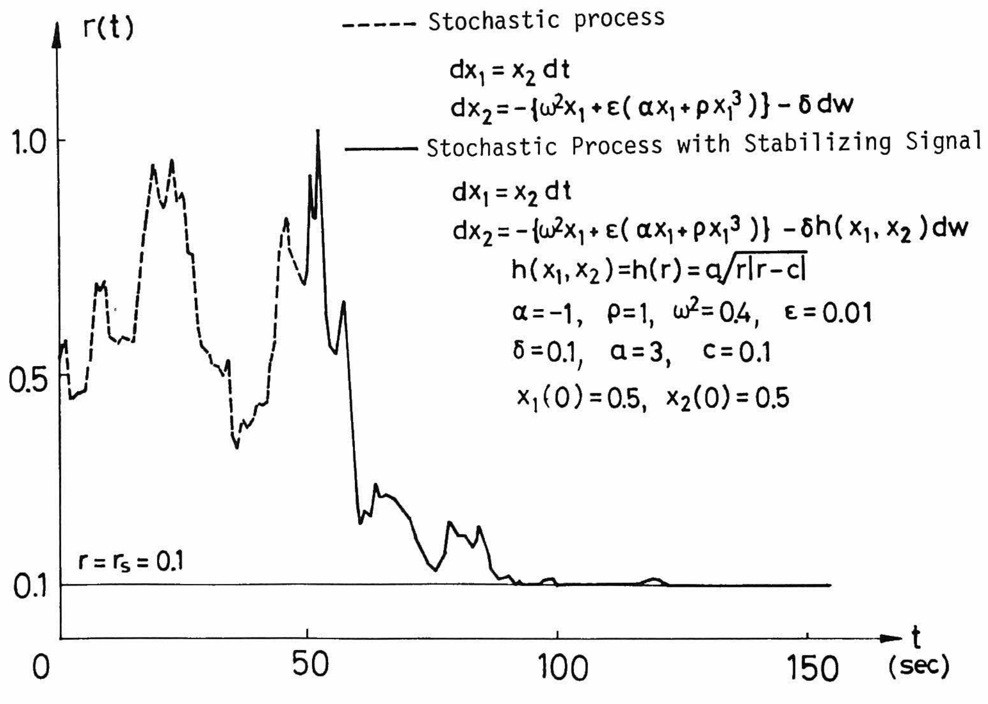


Fig.6.9 A Stabilized Sample Path Behavior of Nonlinear Dynamical Systems Eq.(6.81) by applying the Stabilizing Noise Term Eq.(6.59)

$r_0=0.54$ determined by Eq.(6.78). At time t=50(sec), the stabilizing signal was added , expressed by the second term of the right-hand side of Eq.(6.81). Evidently, the r(t)-process is stabilized and converges to a limit cycle with the smaller amplitude $r_s=0.1$, as shown by the solid curve in Fig.6.8. In this case, a set of parameter values is shown in Fig.6.8 and, in particular, the singular point is $r_s=c=0.1$.

Now, as described in Section 6.2, the key notion of the theoretical development is that the value of ε should be sufficiently small in order to guarantee the application of the averaging principle. Hence, an expected question is how the stabilizing behavior of nonlinear systems depends on the value of ε. This question is examined on Eq.(6.81) by simulation experiments as shown in Figs.6.8(b) and 6.8(c) where $\varepsilon=0.05$ in Fig.6.8(b) and $\varepsilon=0.1$ in Fig.6.8(c). It is reasonably concluded that the smaller value of ε gives a pleasant behavior of stabilization.

As stated in 6.2, noise stabilization of the unstable nonlinear system described by

$$(6.82a)\quad dx_1 = x_2 dt$$

$$(6.82b)\quad dx_2 = -\{\omega^2 x_1 + \varepsilon(\alpha x_1 + \rho x_1^3)\} - \delta dw$$

was also examined by simulation experiments. The result is shown in Fig.6.9. A set of parameter values was set as $\alpha=-1$, $\rho=1$, $\omega^2=0.4$, $\varepsilon=0.01$ and $\delta=0.1$. In Fig.6.9, the dotted curve gives the unstable sample path behavior of the r(t)-process determined by Eq.(6.82) with the initial condition $r(0) \cong 0.59$. At time t=50(sec), instead of δdw in Eq.(6.82), the stabilizing noise term $\delta h(x_1,x_2)dw$ was applied to Eq.(6.82) with the form of Eq.(6.59) where a=3.0 and c=0.1. The solid curve shows the stabilized sample path behavior which converges to the trap r_s=c=0.1.

6.7 Summary

A method of noise stabilization for second order nonlinear dynamical systems has been developed. On the basis of stability criteria established, two possible types of noise terms have, intuitively, been found out; one is biased sinusoidal signal and another biased polynomial type signal. Throughout these studies mentioned above, it was clarified that the selection of stabilizing signals depends on both the nonlinear system characteristics $g(x,\dot{x})$, and the stabilizing function $h(x,\dot{x})$. In examples considered here, there were excellent agreements between the theoretical aspects and the results of digital simulation experiments.

Although the study in this chapter is limited to the stabilization for a specified nonlinear dynamical systems described by Eq.(6.2), the noise stabilization technique developed above will be a guide to the stabilization of other types of nonlinear systems.

Chapter 7

General Conditions for Noise Stabilization of Nonlinear Dynamical Systems

7.1 Introduction

We shall consider again the second order nonlinear stochastic differential equation,

$$(7.1)\qquad \ddot{x} + \omega^2 x + \varepsilon g(x,\dot{x}) = -\delta h(x,\dot{x})\dot{\xi}(t)$$

which was already treated in Chapter 6.

The motivation of the present chapter for noise stabilization may be outlined as follows:

(1) System identification from control point of view

Considering the case where $h(x,\dot{x})=1$, Eq.(7.1) expresses the mathematical model of unstable system subjected to a white Gaussian random input. Consequently, as shown in Fig.7.1, the problem is to design a compensator h which makes the total system stable under given noise condition.

(2) System stabilization from the viewpoint of existing unstable dynamical systems

Considering an unstable nonlinear dynamical system which operates in random environment, the purpose of the present study is to investigate the stabilizing conditions due to environmental noise parameters. Thus it may clearly be understood that the noise stabilization technique is more attractive in the practical aspect of applications than the sinusoidal signal stabilization one.

From viewpoints of (1) and (2) described above, a method of noise stabilization has already been developed in Chapter 6 for nonlinear dynamical systems of Eq.(7.1). The principal line of attack is either to choose intuitively the stabilizing noise term $h(x,\dot{x})$ for $g(x,\dot{x})$ determined already or to examine the possibility of stabilization on the existing term $h(x,\dot{x})\dot{\xi}(t)$. However, this method can never give the whole aspect with respect to the possibility of noise stabilization, though it may be applied to piece-

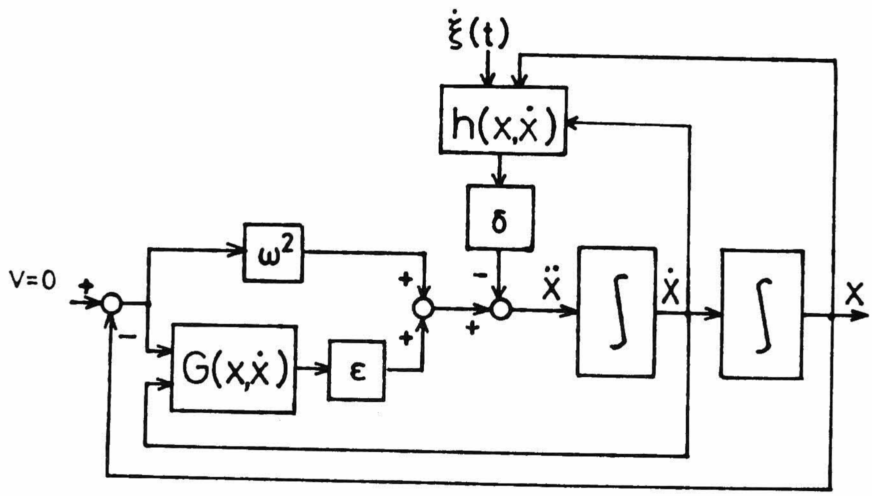


Fig.7.1 Block Diagram of a Control System with a Noise-modulation Stabilizer.

wise examples. Then, in this chapter, a general rule has been established to realize noise stabilization of unstable nonlinear dynamical systems.

7.2 Problem Statement

As the basic equations were already derived in detail in Sections 6.2 and 6.3, we shall review only the description which becomes necessary in understanding this chapter.

Instead of the rectangular coordinate $(x,\dot{x})$ in (7.1), introducing the polar coordinate (r,ψ) along the relation; $x=(r/\omega)\times\sin(\psi-\omega t)$ and $\dot{x}=-r\cos(\psi-\omega t)$ and furthermore applying Khas'minskii's principle of averaging, the differential generator L_r of (7.1) was obtained by

$$(7.2) \qquad L_r = U^2(r)\frac{d^2}{dr^2} + V(r)\frac{d}{dr} ,$$

where

$$(7.3) \qquad U^2(r) = \frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\sin^2\theta d\theta ,$$

$$(7.4) \qquad V(r) = \frac{\sigma^2}{4\pi}\int_0^{2\pi} h^2(\frac{r}{\omega}\cos\theta, r\sin\theta)\frac{\cos^2\theta}{r}d\theta + \Phi(r)$$

and where $\Phi(r)$ is given by Eq.(6.14d) in Section 6.3.

In order to establish a general rule of noise stabilization, a principal line of attack is to apply the Feller's classification criteria of boundaries. Here, especially both the coefficients (7.3) and (7.4) in Eq.(7.2) will play an important role to the noise stabilization problem. The interesting situations arise when the diffusion is singular for which, in Eq.(7.3), the relation $U^2(r=r_s)=0$ holds. Then, the singularity configulations are out-

-lined at or near singularities. To do this, we shall clarify the boundaries of the interval $I=[r_1,r_2]$ formed by singularities on the r-direction at r_1 and r_2, introducing the canonical speed and scale measures defined by Eqs.(2.18) and (2.19) and furthermore using the properties of a pair (σ_i,μ_i) in 2.1.C. Through these studies, the general rule for noise stabilization will be established to clarify how the stabilizing function $h(x,\dot{x})$ should be selected corresponding to any properties of the nonlinear system characteristics $g(x,\dot{x})$ in Eq.(7.1).

7.3 Derivation of General Conditions

Before proceeding to show direct applications of the mathematical classification rule described in the preceding section, sample path behaviors are theoretically examined in the neighborhood around a singularity.

Although an excellent classification rule has been reported in Reference[87] from the viewpoint of noise stabilization, our attention is placed on the function $h(r)$. We shall assume here that a choice of the function h is performed so as to be independent of θ.

This choice allows us to write

$$(7.5)\qquad U^2(r) = \frac{\sigma^2}{4} h^2(r)$$

and

$$(7.6)\qquad V(r) = \frac{\sigma^2}{4r} h^2(r) + \Phi(r).$$

The functions $h^2(r)$ and $\Phi(r)$ may respectively be expanded around a singular point $r=r_s$ into

$$(7.7)\qquad h^2(r) = h^2(r_s) + h^{2\prime}(r_s)(r-r_s) + \frac{1}{2}h^{2\prime\prime}(r_s)(r-r_s)^2 + \cdots$$

and

$$(7.8) \qquad \Phi(r) = \Phi(r_s) + \Phi'(r_s)(r-r_s) + \frac{1}{2}\Phi''(r_s)(r-r_s)^2 + \cdots$$

where "'" denotes the derivative with respect to r. Substituting (7.7) and (7.8) into (7.5) and (7.6), we have

$$(7.9) \qquad U^2(r) = \frac{\sigma^2}{4}\{h^{2'}(r_s)(r-r_s) + \frac{1}{2}h^{2''}(r_s)(r-r_s)^2\}$$

and

$$(7.10) \qquad V(r) = \frac{\sigma^2}{4r}\{h^{2'}(r_s)(r-r_s) + \frac{1}{2}h^{2''}(r_s)(r-r_s)^2\}$$

$$+ \{\Phi(r_s) + \Phi'(r_s)(r-r_s) + \frac{1}{2}\Phi''(r_s)(r-r_s)^2\}$$

where higher order terms than $(r-r_s)^3$ in the expansions (7.7) and (7.8) have been deleted. From (2.14) and (7.5), $h^2(r_s)=0$.

Using (7.9) and (7.10), the function (2.17) may be computed as

$$(7.11) \qquad B_s(r) = \log\frac{r}{r_o} + \frac{4}{\sigma^2}\frac{\Phi''(r_s)(r-r_s)}{h^{2''}(r_s)}$$

$$+ \frac{8}{\sigma^2}\frac{h^{2''}(r_s)\Phi'(r_s)-h^{2'}(r_s)\Phi''(r_s)}{\{h^{2''}(r_s)\}^2}\log|h^{2'}(r_s)+\frac{1}{2}h^{2''}(r_s)(r-r_s)|$$

$$+ \frac{4\Phi(r_s)}{\sigma^2 h^{2'}(r_s)}\log\left|\frac{r-r_s}{h^{2'}(r_s)+(h^{2''}(r_s)/2)(r-r_s)}\right| - (A_o+B_o)$$

where

$$(7.12) \qquad A_o = \frac{4\Phi''(r_s)(r_0-r_s)}{\sigma^2 h^{2''}(r_s)} + \frac{8}{\sigma^2}\frac{h^{2''}(r_s)\Phi'(r_s)-h^{2'}(r_s)\Phi''(r_s)}{\{h^{2''}(r_s)\}^2}$$

$$\times\log|h^{2'}(r_s) + \frac{1}{2}h^{2''}(r_s)(r_0-r_s)|$$

and

$$(7.13)\qquad B_0 = \frac{4\Phi(r_s)}{\sigma^2 h^{2'}(r_s)}\,\ell og\left|\frac{r_0-r_s}{h^{2'}(r_s)+(h^{2''}(r_s)/2)(r_0-r_s)}\right| .$$

With Eq.(7.11), Eqs.(2.18) and (2.19) become,

$$(7.14)\qquad dm(r) = z_m dr$$

and

$$(7.15)\qquad ds(r) = z_s dr$$

where

$$(7.16)\qquad z_m = \frac{4}{A_c r_0 \sigma^2} r R^{\beta-\gamma-1}(r-r_s)^{\gamma-1}\exp\left\{\frac{4}{\sigma^2}\frac{\Phi'(r_s)(r-r_s)}{h^{2''}(r_s)}\right\}$$

and

$$(7.17)\qquad z_s = \frac{A_c r_0}{r} R^{-\beta+\gamma}(r-r_s)^{-\gamma}\exp\left\{-\frac{4}{\sigma^2}\frac{\Phi''(r_s)(r-r_s)}{h^{2''}(r_s)}\right\}.$$

In Eqs.(7.16) and (7.17), the parameters are given by

$$(7.18)\qquad R = h^{2'}(r_s) + (h^{2''}(r_s)/2)(r-r_s),\quad A_c = \exp(A_0+B_0)$$

$$(7.19)\qquad \beta = \frac{8}{\sigma^2}\frac{h^{2''}(r_s)\Phi'(r_s)-h^{2'}(r_s)\Phi''(r_s)}{\{h^{2''}(r_s)\}^2} \text{ and } \gamma = \frac{4}{\sigma^2}\frac{\Phi(r_s)}{h^{2'}(r_s)} .$$

Using (7.14) and (7.15), we shall proceed to make the general classification of the boundaries.

7.3.A <u>The case where a singular point $r=r_s$ is the trap</u> (In Eq.(7.2), $U^2(r_s)=0$ and $V(r_s)=0$)

(A) $h^{2'}(r_s) \neq 0$

In this case, from Definition 2-1, (7.6) and (7.8), it is apparent that $\Phi(r_s)=0$ or, from (7.19), $\gamma=0$. Hence, Eqs.(7.16) and (7.17) are respectively

$$(7.20)\qquad z_m = \frac{4rR^{\beta-1}}{A_c r_0 \sigma^2}(r-r_s)^{-1}\exp\{\frac{4}{\sigma^2}\frac{\Phi''(r_s)(r-r_s)}{h^{2''}(r_s)}\}$$

and

$$(7.21)\qquad z_s = \frac{A_c r_0}{r}R^{-\beta}\exp\{-\frac{4}{\sigma^2}\frac{\Phi''(r_s)(r-r_s)}{h^{2''}(r_s)}\}.$$

In order to classify the boundaries, using Eqs.(7.20) and (7.21), Eqs.(2.20a) and (2.20b) are respectively examined in the forms,

$$(7.22)\qquad \sigma(r) = \int_r^{r_1} z_s(r_y)dr_y \int_{r_y}^{r_1} z_m(r_x)dr_x$$

and

$$(7.23)\qquad \mu(r) = \int_r^{r_1} z_m(r_y)dr_y \int_{r_y}^{r_1} z_s(r_x)dr_x,$$

where r_1 is a constant. A general rule can be obtained by applying Eqs.(7.22) and (7.23) to properties of a pair (σ,μ) described in Section 2.1.C.

(B) $h^{2'}(r_s) = 0$

Letting $h^{2'}(r_s)=0$, (2.17) may be computed as

$$(7.24)\qquad B_s(r) = \log\frac{r}{r_0} + \frac{4}{\sigma^2}\frac{\Phi''(r_s)}{h^{2''}(r_s)}r + \frac{8}{\sigma^2}\frac{\Phi'(r_s)}{h^{2''}(r_s)}\log|r-r_s| - (A_{01}+B_{01})$$

where

$$(7.25)\qquad A_{01} = \frac{4}{\sigma^2}\frac{\Phi''(r_s)}{h^{2''}(r_s)}r_0 \quad\text{and}\quad B_{01} = \frac{8}{\sigma^2}\frac{\Phi'(r_s)}{h^{2''}(r_s)}\log|r_0-r_s|.$$

Hence, we have

$$(7.26)\qquad z_{m1} = \frac{8}{\sigma^2 A_{01} r_0 h^{2''}(r_s)}r(r-r_s)^{\beta_1-2}\exp\{\frac{4}{\sigma^2}\frac{\Phi''(r_s)}{h^{2''}(r_s)}r\}$$

and

$$(7.27) \qquad z_{s1} = \frac{A_{01} r_0}{r}(r-r_s)^{-\beta_1} \exp\{-\frac{4\Phi''(r_s)}{\sigma^2 h^{2''}(r_s)} r\}$$

where

$$(7.28) \qquad \beta_1 = 8\Phi'(r_s)/\sigma^2 h^{2''}(r_s), \qquad A_{01} = \exp(A_{01}+B_{01}).$$

Instead of z_m and z_s given by (7.20) and (7.21) respectively, using (7.26) and (7.27) to (7.22) and (7.23), classification of the boundaries are made. The results are summarized in Table 7.1.

Table 7.1 Classification of the trap

No.	Conditions of coefficients		σ	μ	Classification
1	$h^2(r_s)=0, r_s\neq 0$ $h^{2'}(r_s)\neq 0$ $h^{2''}(r_s)\neq 0$ or $=0$		$<\infty$	∞	accessible exit boundary
2	$h^2(r_s)=0, r_s=0$ $h^{2'}(r_s)\neq 0$ $h^{2''}(r_s)\neq 0$ or $=0$		∞	$<\infty$	inaccessible entrance boundary
3	$h^2(r_s)=0$ $h^{2'}(r_s)=0$ $h^{2''}(r_s)\neq 0$ $r_s\neq 0$	$\beta_1\geq 1$ $s(r_s)=\infty$	∞	∞	inaccessible natural boundary (locally unattractive)
		$\beta_1\leq 1/2$ $s(r_s)<\infty$	∞	∞	inaccessible natural boundary (locally attractive)
4	$h^2(r_s)=0$ $h^{2'}(r_s)=0$ $h^{2''}(r_s)\neq 0$ $r_s=0$	$\beta_1\geq 0$ $s(0)=\infty$	∞	∞	inaccessible natural boundary (locally unattractive)
		$\beta_1\leq -1/2$ $s(0)<\infty$	∞	∞	inaccessible natural boundary (locally attractive)

7.3.B <u>The case where a singular point $r=r_s$ is not the trap</u> (In Eq.(7.2), $U^2(r_s)=0$ and $V(r_s)\neq 0$)

(A) $h^{2'}(r_s) \neq 0$, $\gamma(\neq 0) < \infty$

In this case, since $\Phi(r_s)\neq 0$, z_m and z_s become respectively

$$z_m = \frac{4rR^{\beta-\gamma-1}}{A_0 r_0 \sigma^2}(r-r_s)^{\gamma-1}\exp\{\frac{4}{\sigma^2}\frac{\Phi''(r_s)(r-r_s)}{h^{2''}(r_s)}\} \tag{7.29}$$

$$z_s = \frac{A_0 r_0}{r}R^{-\beta+\gamma}(r-r_s)^{-\gamma}\exp\{-\frac{4}{\sigma^2}\frac{\Phi''(r_s)(r-r_s)}{h^{2''}(r_s)}\}. \tag{7.30}$$

Classifications of the boundaries are made by (7.29) and (7.30) to (7.22) and (7.23) as shown in Table 7.2.

<u>Table 7.2 Classification of the boundaries in the case where r_s is a singular point (I) ($h^{2'}(r_s)\neq 0$)</u>

No.	Conditions of coefficients*		σ	μ	Classification
1	$h^2(r_s)=0$, $h^{2'}(r_s)\neq 0$, $h^{2''}(r_s)\neq 0$ or $=0$ for all β, $r_s\neq 0$	$\gamma\geq 1$	∞	$<\infty$	inaccessible entrance boundary
2		$\gamma=\frac{1}{2}$	$<\infty$	$<\infty$	accessible regular boundary
3		$\gamma\leq 0$	$<\infty$	∞	accessible exit boundary
4	$h^2(r_s)=0$, $h^{2'}(r_s)\neq 0$, $h^{2''}(r_s)\neq 0$ or $=0$ for all β, $r_s=0$	$\gamma\geq 0$	∞	$<\infty$	inaccessible entrance boundary
5		$\gamma=-\frac{1}{2}$	$<\infty$	$<\infty$	accessible regular boundary
6		$\gamma\leq -1$	$<\infty$	∞	accessible exit boundary

* It is extremely difficult to classify the boundaries with respect to $1/2<\gamma<1$, $0<\gamma<1/2$, $-1/2<\gamma<0$ and $-1<\gamma<-1/2$.

(B) $h^{2'}(r_s) = 0$, $\gamma = \infty$

In this case, the mathematical situation is somewhat complicated. Letting $h^{2'}(r_s)=0$ in (7.9) and (7.10), instead of (7.11), we have

$$(7.31) \qquad B_s(r) = \log\frac{r}{r_0} + \frac{8}{\sigma^2 h^{2''}(r_s)}\{\frac{-\Phi(r_s)}{r-r_s} + \Phi'(r_s)\log|r-r_s| + \frac{1}{2}\Phi''(r_s)r\} - c_0$$

where

$$(7.32) \qquad c_0 = \frac{8}{\sigma^2 h^{2''}(r_s)}\{\frac{-\Phi(r_s)}{r_0-r_s} + \Phi'(r_s)\log|r_0-r_s| + \frac{1}{2}\Phi''(r_s)r_0\}.$$

Hence, it follows that

Table 7.3 Classification of the boundaries in the case where r_s is a singular point (II) ($h^{2'}(r_s)=0$)

No.	Conditions of coefficients		σ	μ	Classification
1	$h^2(r_s)=0$, $h^{2'}(r_s)=0$	$\eta>0$, $s(r_s)=\infty$	∞	∞	inaccessible natural boundary (locally unattractive)
2	$h^{2''}(r_s)\neq 0$, $\Phi(r_s)\neq 0$	$\eta<0$, $s(r_s)<\infty$	∞	∞	inaccessible natural boundary (locally attractive)

(Note) $\eta=8\Phi(r_s)/\sigma^2 h^{2''}(r_s)$

$$(7.33)\qquad z_m = \frac{8}{\sigma^2 c_0 r_0 h^{2''}(r_s)} r(r-r_s)^{\beta_1-2}\exp\{\frac{8}{\sigma^2 h^{2''}(r_s)}(\frac{-\Phi(r_s)}{r-r_s} + \frac{1}{2}\Phi''(r_s)r)\}$$

and

$$(7.34)\qquad z_s = \frac{c_0 r_0}{r}(r-r_s)^{-\beta_1}\exp\{-\frac{8}{\sigma^2 h^{2''}(r_s)}(\frac{-\Phi(r_s)}{r-r_s} + \frac{1}{2}\Phi''(r_s)r)\}.$$

Combining (7.33) and (7.34) with (7.22) and (7.23), classification of the boundaries is summarized as shown in Table 7.3.

7.4 Examples of Noise Stabilization

General considerations developed in the previous section are applied to realize the noise stabilization of unstable nonlinear dynamical systems.

7.4.A Nonlinear Dynamical System of Van der Pol-type with Polynomial Stabilizing Term

As the first example, we shall consider the dynamical system whose sample path is determined by

$$(7.35)\qquad \begin{aligned} dx_1 &= x_2 dt \\ dx_2 &= -\{x_1-\varepsilon(1-x_1^2)x_2\}dt - \delta h(x_1,x_2)dw. \end{aligned}$$

A choice is made on $h(x_1,x_2)$ of the form,

$$(7.36)\qquad h^2(r) = r^2|r^2-c^2|$$

where c is an arbitrary constant.

From (6.14d) in 6.3, (7.3) and (7.4), we have

$$(7.37)\qquad \Phi(r) = \frac{\sigma^2 r}{4}(2-\frac{r^2}{2}), \qquad (r\geq c\geq 0)$$

(7.38) $U^2(r) = \frac{\sigma^2 r^2}{4}(r^2-c^2)$, $\quad (r \geq c > 0)$

and

(7.39) $V(r) = \frac{\sigma^2 r}{4}(r^2-c^2) + \frac{r}{4}(2-\frac{r^2}{2})$.

Set as $U^2(r)=0$. Then, we have two singular points, i.e., $r_{s1}=0, r_{s2}=c$.

(i) The singular point $r_{s1}=0$: From (7.39), it is apparent that since $V(0)=0$, the origin is the trap.

(ii) The singular point $r_{s2}=c$: Apparently, the point $r_{s2}=c=2$ is the trap provided that $c=2$. If $c\neq 2$, then r_{s2} is a singular point which is not the trap.

From (7.7) and (7.36), it follows that

(7.40) $h^{2'}(r) = 2r(2r^2-c^2)$

and

(7.41) $h^{2''}(r) = 2(6r^2-c^2)$.

(A) The case where $c\neq 2$.

(a) The interval (c,∞): In this case, (7.36) is rewritten as

(7.42) $h^2(r) = r^2(r^2-c^2)$.

From (7.40) and (7.41), we have $h^{2'}(c)=2c^3\neq 0$, $h^{2''}(c)=10c^2\neq 0$. Also, since

(7.43) $\gamma = 4\Phi(c)/\sigma^2 h^{2'}(c) = (4-c^2)/4c^2\sigma^2$,

we have that $\gamma<0$ for $c>2$ and $\gamma>0$ for $c<2$. Thus, according to the general rules in Table 7.2, it is concluded that

[I] $r_s=c$ $(c>2)$ is the accessible exit boundary

[II] $r_s=c$ $(c<2)$ is the accessible regular boundary provided that $\sigma^2 =(4-c^2)/2c^2$ and if $\sigma^2\leq(4-c^2)/4c^2$, then $r_s=c$ is the inaccessible entrance boundary.

On the other hand, it follows that

(7.44) $$dm(r) = \frac{4r_0}{r}\left(\frac{c}{r}\right)^2 \frac{(r_0^2-c^2)^{(c^2-4)/4}}{(r^2-c^2)^{c^2/4}} dr .$$

and

(7.45) $$ds(r) = \frac{r}{r_0}\left(\frac{r_0^2-c^2}{r^2-c^2}\right)^{(2-c^2/2)/2} dr.$$

It is a simple exercise to show that $\sigma_{r\to\infty}=+\infty$ and $\mu_{r\to\infty}<\infty$ and to conclude that the boundary $r=\infty$ is the inaccessible entrance boundary.

(b) The interval $(0,c)$: In this case, since (7.36) is

(7.46) $$h^2(r) = r^2(c^2-r^2),$$

we have $h^{2'}(c)=-2c^3 \neq 0$ and $h^{2''}(c)=-10c^2 \neq 0$. Furthermore, by a similar consideration to (7.43), we have conclusions $\gamma<0$ for $c<2$ and $\gamma>0$ for $c>2$. Thus, it is concluded that

[III] $r_s=c(c>2)$ is the accessible regular boundary, provided that $\sigma^2=(4-c^2)/2c^2$. If $\sigma^2 \leq (4-c^2)/4c^2$, then $r_s=c$ is the inaccessible entrance boundary.

[IV] $r_s=c(c<2)$ is the accessible exit boundary.

On the other hand, it is obvious that $h^2(0)=0$, $h^{2'}(0)=0$ and $h^{2''}(0) \neq 0$. With a computation of (7.19), i.e.,

(7.47) $$\beta = \frac{2}{\sigma^2 c^2} > 0$$

and with the application of general rule to the present example, the trap $r=0$ is inaccessible natural boundary (locally unattractive).

(B) The case where $c=2$

(a) The interval $(2,\infty)$: Noting that (7.36) is given by

$$(7.48) \qquad h^2(r) = r^2(r^2-4),$$

it follows that $h^{2'}(2)\neq 0$ and $h^{2''}(2)\neq 0$. Thus $r=0$ is the accessible exit boundary. (attractive trap) Furthermore, using the same procedure as in the case (A), it is easily concluded that $r=\infty$ is the inaccessible entrance boundary.

(b) The interval $(0,2)$: Noting that $h^2(r)=r^2(4-r^2)$, $h^{2'}(2)\neq 0$ and $h^{2''}(0)\neq 0$, from Table 7.1, it may be concluded that $r=2$ is the

Table 7.4 Classification of the boundaries in Example-1

Interval	Value of c	σ	μ	Classification of the boundaries
$r=0$	——	∞	∞	inaccessible natural boundary (locally unattractive)
↓	$c<2$	$<\infty$	∞	accessible exit boundary
	$c=2$	$<\infty$	∞	accessible exit boundary
$r=c$	$c>2$	$<\infty$	$<\infty$	accessible regular boundary
$r=c$	$c<2$	$<\infty$	$<\infty$	accessible regular boundary
↓	$c=2$	$<\infty$	∞	accessible exit boundary
	$c>2$	$<\infty$	∞	accessible exit boundary
$r=\infty$	——	∞	$<\infty$	inaccessible entrance boundary

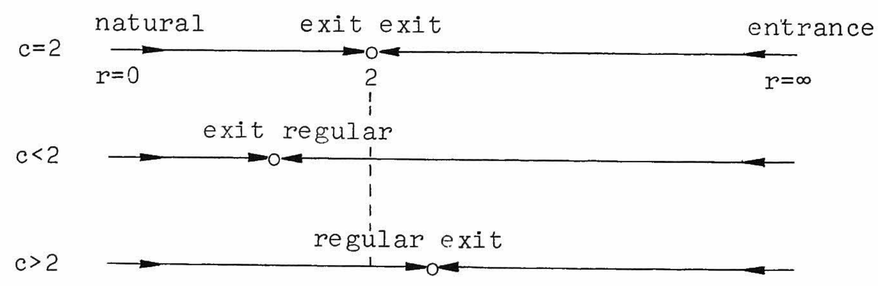


Fig 7.2 Sample Path Behaviors of the System described by Eq.(7.35)

exit boundary. The origin $r=0$ is the inaccessible natural boundary (unattractive trap). This is easily concluded by the same procedure as in the case (A). The results of classification of the boundaries are summarized in Table 7.4. Also, based on the results of Table 7.4, the sample path behaviors of the system described by Eq.(7.35) are shown in Fig.7.2.

As already pointed out in (ii), the point $r_{s2}=c=2$ is the trap where $c=2$ for which $V(2)=0$ and, from (7.6), $\Phi(r)|_{r=c=2}=0$. It can be easily observed that the point $r=c=2$ shows the stable limit cycle of the system modeled by

$$\ddot{x} + x + \varepsilon g(x,\dot{x}) = 0 \tag{7.49a}$$

where

$$g(x,\dot{x}) = -(1-x^2)\dot{x} \tag{7.49b}$$

and, for convenience of discussions, a simple case where $\beta=1$ and $\omega=1$ has been considered. According to Ref.[77], the condition of existence of the stable limit cycle of the system (7.49) is as follows.

Let r_i be a root of $\Phi(r)=0$. Then,

(1^o) if $d\Phi(r_i)/dr_i<0$, then the stable limit cycle exists at $r=r_i$.

(2^o) if $d\Phi(r_i)/dr_i>0$, then the limit cycle at $r=r_i$ is unstable.

Since, from (7.6) and (7.49b),

$$\Phi(r) = \frac{r}{2}(1-\frac{r^2}{4}) \tag{7.50}$$

and $\Phi'(r)|_{r=2}<0$, the limit cycle at $r=2$ is stable. Thus, in the choice of the function h as given by (7.42), the value of c such that $V(r)=0$ gives the stable limit cycle without noisy background.

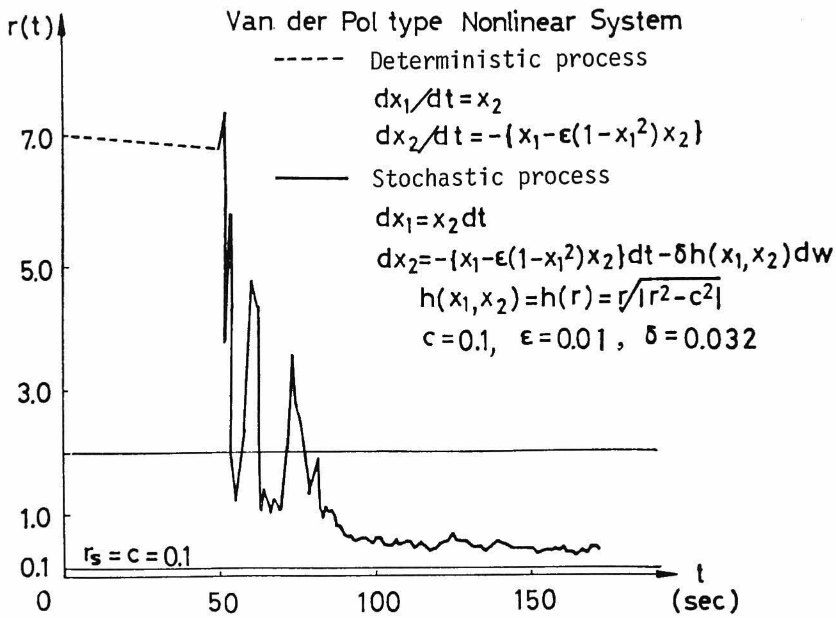


Fig.7.3 A Stabilized Sample Path Behavior of Nonlinear Dynamical System described by Eq.(7.35)

Figure 7.3 shows the sample path behavior of the system (7.35) with the stabilizing noise term of the form (7.36). Simulation experiments were performed by a similar procedure as described in Section 6.6. A set of parameter values is preassigned as $c=0.1$, $\varepsilon=0.01$, $\delta=0.032$. The dotted line represents the deterministic behavior starting with the initial value $r(0)=7.0$. At time $t=50$(sec), the stabilizing noise signal was applied to the system, whose sample path was shown by the solid curve. It can be easily understood that the system was stabilized by applying the stabilizing noise and that the $r(t)$-process converges to the trap $r_s = 0.1$. This implies the sample path behavior in the case where $c < 2$ in Fig.7.2.

7.4.B Nonlinear Dynamical System of Froude-type with Polynomial Stabilizing Noise

Nextly, we shall consider the dynamical system

(7.51a) $dx_1 = x_2 dt$

(7.51b) $dx_2 = -\{x_1 + \varepsilon(\alpha x_2 + \rho x_2^2 - \eta x_2^3)\}dt - \delta h(x_1, x_2)dw$

with the stabilizing noise term given by (7.36), where α, ρ and η are constants and $\alpha>0$, $\rho<0$, $\eta>0$. From (6.14d) in 6.3, (7.9),(7.10) and (7.51), we have

(7.52) $\Phi(r) = \frac{3\eta}{8} r\,(r^2 - \frac{4\alpha}{3\eta})$

(7.53) $U^2(r) = \frac{r^2}{4}|r^2 - c^2|$

and

(7.54) $V(r) = \frac{r}{4}|r^2 - c^2| + \frac{3\eta r}{8}(r^2 - \frac{4\alpha}{3\eta})$.

Letting $U^2(r)=0$, then we have two singular points, i.e., $r_{s1}=0$ and $r_{s2}=c$.

(i) $r_{s1}=0$: From (7.54), since $V(0)=0$, the origin is the trap.

(ii) $r_{s2}=c$: The point $r_{s2}=c$ is the trap, provided that $c=\sqrt{4\alpha/3\eta}$. If $c\neq\sqrt{4\alpha/3\eta}$, then r_{s2} is not the trap for which further examination is required.

(A) The case where $c \neq \sqrt{4\alpha/3\eta}$

(a) The interval (c,∞) : From (7.36), it follows that $h^{2'}(c)\neq0$ and $h^{2''}(c)\neq0$. Since, from (7.19),

(7.55) $\gamma = (3\eta c^2 - 4\alpha)/4\sigma^2 c^2$,

we have the conclusions that $\gamma<0$ for $c < \sqrt{4\alpha/3\eta}$ and $\gamma>0$ for $c >$

$\sqrt{4\alpha/3\eta}$. Then, following Table 7.2,

[I] $r_{s2} = c < \sqrt{4\alpha/3\eta}$ is the accessible exit boundary.

[II] $r_{s2} = c > \sqrt{4\alpha/3\eta}$ is the accessible regular boundary, provided that $\sigma^2 = 2(4\alpha - 3\eta c^2)/c^2$ and if $\sigma^2 \leq (4\alpha - 3\eta c^2)/c^2$, then $r_{s2}=c$ is the inaccessible entrance boundary.

We shall proceed to classify the sample path behavior at $r\to\infty$. From (7.52), (7.53) and (7.54), the calculations of (7.14) and (7.15) become,

(7.56) $$dm(r) = \frac{4c^2}{\alpha}\left(\frac{r}{r_0}\right)^{-1+2\zeta}\left(\frac{r^2-c^2}{r_0^2-c^2}\right)^{\zeta(\lambda-2)} dr$$

and

(7.57) $$ds(r) = \left(\frac{r^2-c^2}{r_0^2-c^2}\right)^{\zeta(1-\lambda)} r^{-1-2\zeta} dr,$$

where

(7.58) $\lambda = 3\eta c^2/4\alpha$ and $\zeta = \alpha/\sigma^2 c^2$.

From the viewpoint of noise stabilization, the value of c must be small so that we shall assume here that $0<\lambda<1$ and $\zeta<1$. With the results of computations $\sigma_{r=\infty} = \infty$ and $\mu_{r=\infty} < \infty$, it may easily be concluded that $r \to \infty$ is the inaccessible entrance boundary.

(b) The interval (0,c) : In this case, noting that $h^2(r)=r^2(c^2-r^2)$ and that $h^{2'}(r)\neq 0$ and $h^{2''}(r)\neq 0$, it is concluded that

[III] $r_{s2} = c > \sqrt{4\alpha/3\eta}$ is the accessible exit boundary.

[IV] $r_{s2} = c < \sqrt{4\alpha/3\eta}$ is the accessible regular boundary.

Finally, we shall classify the origin. Since $h^2(0)=0$, $h^{2'}(0)=0$, $h^{2''}(0)\neq 0$ and $\beta>0$, the origin is the inaccessible natural boundary.

(B) The case where $c = \sqrt{4\alpha/3\eta}$

(a) The interval (c,∞) : The stabilizing noise term is given by

$$(7.59) \quad h^2(r) = r^2 (r^2 - \frac{4\alpha}{3\eta}) .$$

Hence, we have $h^{2'}(c) \neq 0$ and $h^{2''}(c) \neq 0$. Thus, according to Table 7.1, it may be concluded that $r_{s2}=c$ is the accessible exit boundary. On the other hand, the conclusion that $r \to \infty$ is the inaccessible entrance boundary is the direct consequence of the case (A).

(b) The interval $(0,c)$: According to the similar procedure mentioned above, it follows that $h^2(c) \neq 0$, $h^{2''}(c) \neq 0$ for $r=c$. Thus, $r_{s2}=c$ is the accessible exit boundary. On the other hand, since $h^{2'}(0)=0$, $h^{2''}(0) \neq 0$ and $\beta > 0$, using Table 7.1, the origin is the inaccessible natural boundary.

The whole aspect of sample path behaviors of the system considered is shown in Fig.7.4. Figure 7.5 shows one of simulation

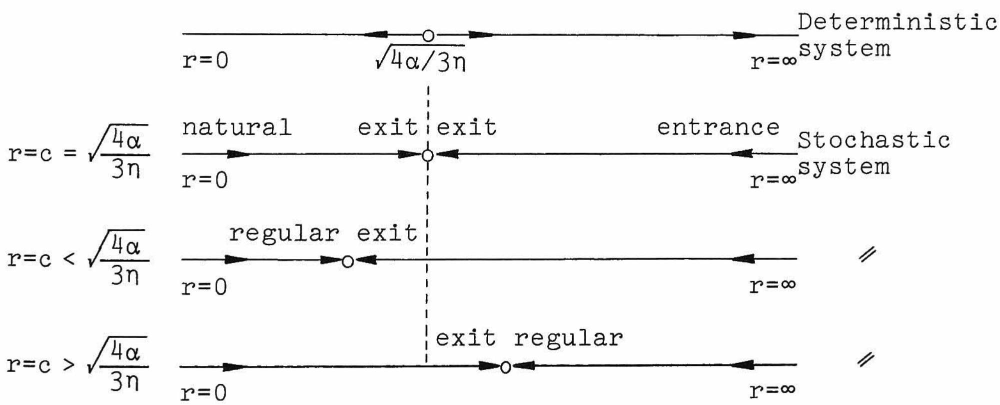


Fig.7.4 The Whole Aspect of Sample Path Behaviors of the System described by Eq.(7.51) (The case where $\alpha > 0$, $\rho < 0$, $\eta > 0$)

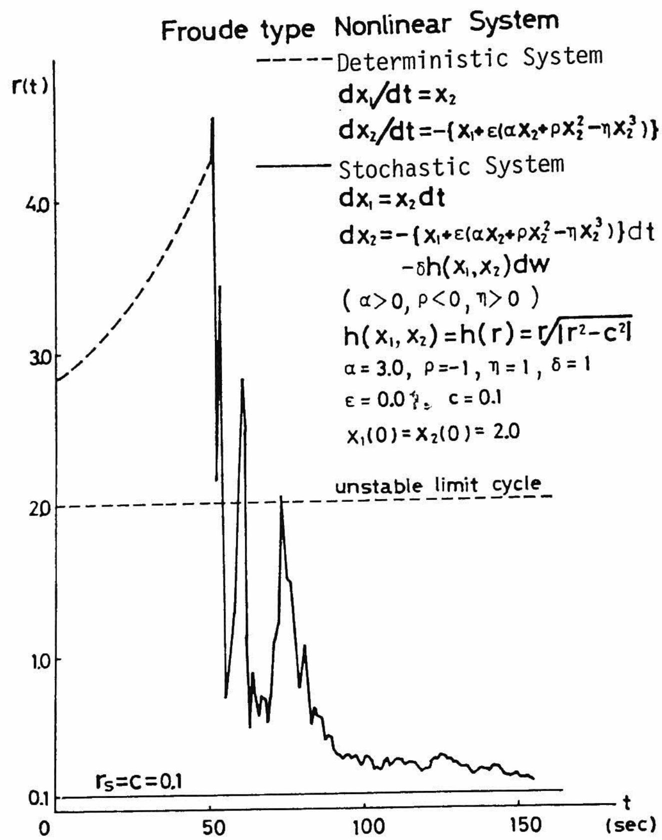


Fig.7.5 A Stabilized Sample Path Behavior of Nonlinear Dynamical System described by Eq.(7.51)

experiments in the case where $c < \sqrt{4\alpha/3\eta}$ in Fig.7.4. A set of system parameters was preassigned as $\alpha=3$, $\rho=-1$, $\eta=1$ and $\varepsilon=0.01$. It is well-known that the system without noisy background is unstable starting with the initial value $r(0)=\sqrt{8}$, as shown by the dotted curve in Fig.7.5 (see Ref.[77]). The stabilizing noise signal (7.36) was applied at t=50(sec) with the values of $\beta=1$ and $c=0.1$. The Stabilizing behavior is shown by the solid curve.

In the case where $\alpha < 0$, $\rho < 0$, $\eta < 0$ in (7.51), the system considered has the stable limit cycle[77]. Consequently, the whole aspect of sample path behaviors of the system differs from that as shown in Fig.7.4. By the same procedure as in Fig.7.4, the boundaries are classified as shown in Fig.7.6. Simulation experiments are also shown in Fig.7.7. With the help of Ref.[77], it can be explored that the system has the limit cycle with the value of r=2.0 under a set of parameter values indicated in Fig.7.7. Figure 7.7 shows the stabilized sample path of the system with the application of stabilizing term given by (7.36) where the initial values of the system state were $x_1(0)=x_2(0)=4.0$, i.e., r(0)=5.65. As shown by the dotted curve, the deterministic behavior without noisy background initiated at r(0)=5.65 approaches to the limit cycle with r=2.0 as time goes on. As usual, the stabilizing noise term was applied at time t=50(sec). The system was stabilized and the sample path of r(t)-process approaches to the singular point $r_s=0.1$

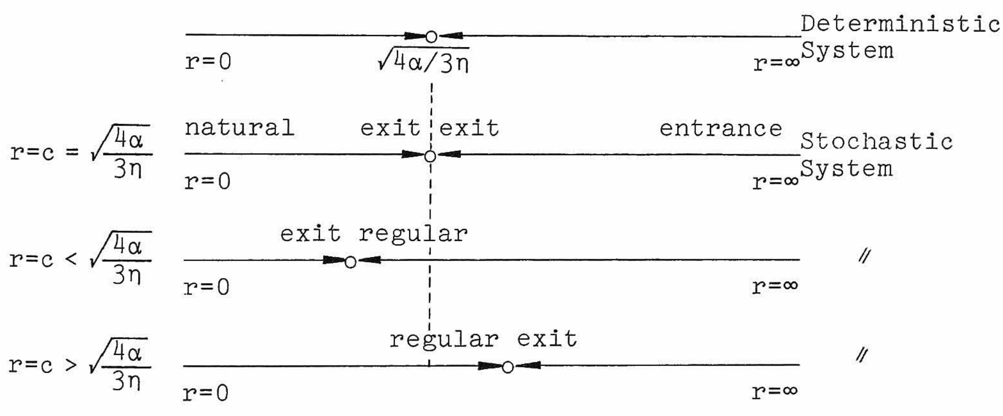


Fig.7.6 The Whole Aspect of Sample Path Behaviors of the System described by Eq.(7.51) (The case where $\alpha < 0$, $\rho < 0$, $\eta < 0$)

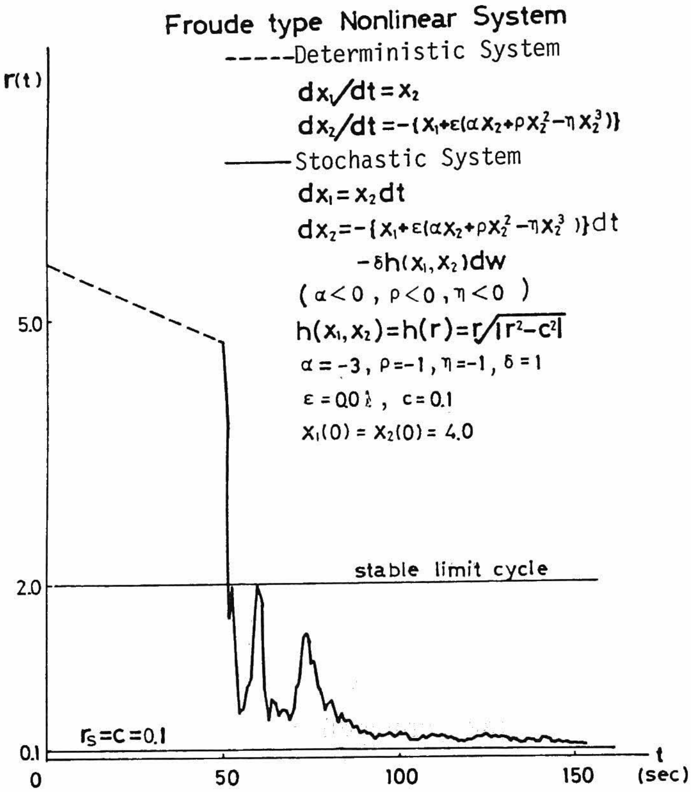


Fig.7.7 A Stabilized Sample Path Behavior of Nonlinear Dynamical System described by Eq.(7.51)

as shown by the solid curve. Conversely, the sample path behavior initiated at the inside of the limit cycle is shown in Fig.7.8. It can be easily concluded that the system was stabilized and the sample path approaches to the singular point $r_s = c = 0.05$.

7.4.C Nonlinear Dynamical System of Duffing-type with Biased-Sinusoidal Stabilizing Noise

Let $g(x_1,x_2)$ in Eq.(7.1) be given by

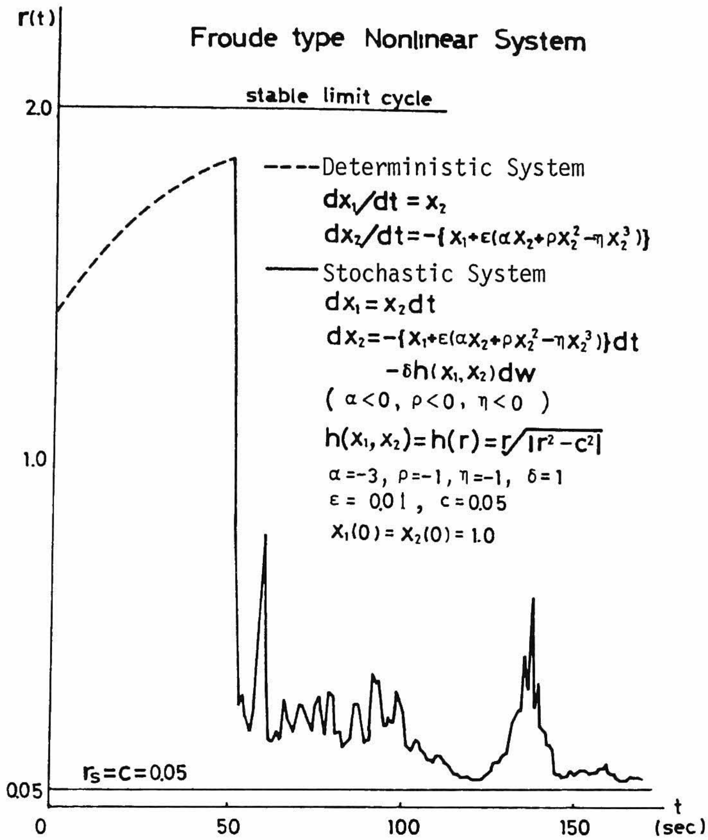


Fig.7.8 A Stabilized Sample Path Behavior of Nonlinear Dynamical System described by Eq.(7.51)

$$(7.60) \qquad g(x,\dot{x}) \equiv g(x_1,x_2) = \alpha x_1 + \rho x_1^3$$

where both α and ρ are arbitrary constants respectively. Also, as the stabilizing noise term, let the function $h(x_1,x_2)$ in Eq.(7.1) be chosen by

$$(7.61) \qquad h(x_1,x_2) = h(r) = a\sqrt{\frac{\cos br}{r} - k}\ ,$$

where, for convenience of description, the polar form has been used and where a, b and k are arbitrary constants respectively. Hence, with (7.60) and (7.61), Eq.(6.4) becomes

$$(7.62a)\quad dx_1 = x_2 dt$$

$$(7.62b)\quad dx_2 = -\{\omega^2 x_1 + \varepsilon(\alpha x_1 + \rho x_1^3)\}dt - \delta a\sqrt{\frac{\cos br}{r} - k}\, dw.$$

Although this example was already treated in Chapter 6, we shall consider again the noise stabilization by applying a general rule developed in the previous section.

Computations of (6.14d) in 6.3, (7.9) and (7.10) by using (7.62) bring

$$(7.63)\quad \Phi(r) = 0,$$

$$(7.64)\quad U^2(r) = \frac{\sigma^2 a^2}{4}\left|\frac{\cos br}{r} - k\right|$$

and

$$(7.65)\quad V(r) = \frac{\sigma^2 a^2}{4r}\left|\frac{\cos br}{r} - k\right|.$$

Thus, the singular point is determined by solving $U^2(r)=0$. In Eq.(7.64), provided that $k \geq 0.16b$ as shown in Fig.6.3, we obtain one singular point. Then, noting that $V(r_s)=0$, the singular point $r=r_s(\neq 0)$ is the trap.

(a) Interval $[r_s,\infty)$: First of all, from (7.7), we have

$$(7.66)\quad h^2(r) = a^2\left|\frac{\cos br_s}{r_s} - k\right| + a^2\left(\frac{b\sin br_s}{r_s} + \frac{\cos br_s}{r_s^2}\right)(r - r_s)$$
$$+ \frac{a^2}{2}\left(\frac{b^2\cos br_s}{r_s} - \frac{2b\sin br_s}{r_s^2} - \frac{2\cos br_s}{r_s^3}\right)(r - r_s)^2 + \cdots$$

Since it is obvious that $h^{2'}(r_s) = 0$, $h^{2''}(r_s) = 0$ from Table 7.1, it is concluded that the point $r=r_s$ is the attractive trap or the accessible exit boundary. Next, we shall examine the case of $r=\infty$. Since $U^2(r)|_{r\to\infty} \neq 0$, this situation does not allow us to apply directly the general rule. The stabilizing noise term h(r) given by (7.61) may be approximated by $h(r)|_{r\to\infty} \cong k$. Thus, from (7.64) and (7.65), $U^2(r)$ and V(r) are approximately expressed by

$$(7.67) \qquad U^2(r)|_{r\to\infty} \cong \frac{\sigma^2 a^2 k}{4}, \qquad V(r)|_{r\to\infty} \cong \frac{\sigma^2 a^2 k}{4r}.$$

Hence, it follows that

$$(7.68) \qquad dm(r) = \frac{4r}{\sigma^2 a^2 k r_0} dr, \qquad ds(r) = \frac{r_0}{r} dr.$$

Applying (7.68) to (7.22) and (7.23), we have

$$(7.69) \qquad \sigma(r) = \lim_{r\to\infty} \int_r^{r_1} \frac{r_0}{r_y} dr_y \int_{r_y}^{r_1} \frac{4}{\sigma^2 a^2 k r_0} r_x dr_x = \infty$$

and

$$(7.70) \qquad \mu(r) = \lim_{r\to\infty} \frac{4}{\sigma^2 a^2 k r_0} \int_r^{r_1} r_y dr_y \int_{r_y}^{r_1} \frac{r_0}{r_x} dr_x = \infty.$$

Furthermore, from (7.68), we have

$$(7.71) \qquad s(r) = \lim_{r\to\infty} \int_r^{r_1} \frac{r_0}{r_x} dr_x = \infty.$$

This fact implies that the point $r\to\infty$ is the inaccessible natural boundary (locally unattractive).

(b) Interval $(0,r_s)$: According to discussion in (a), it is obvious that the singular point $r=r_s$ is the accessible exit boundary.

Instead of (7.61), we shall suppose that, as $r \to 0$, $h(r)$ is approximated by $h(r)=k_0/r$, where k_0 is an arbitrary constant. Using this approximation, from (7.64) and (7.65), it follows that

$$(7.72) \qquad U^2(r) = \frac{\sigma^2 a^2}{4}\frac{k_0}{r}, \qquad V(r) = \frac{\sigma^2 a^2}{4}\frac{k_0}{r^2}.$$

Hence, we have

$$(7.73) \qquad dm(r) = \frac{4r^2}{\sigma^2 a^2 k_0 r_0} dr, \qquad ds(r) = \frac{r_0}{r} dr.$$

It is a simple exercise to conclude that, at $r \to 0$, $\sigma(0)$ is not integrable but $\mu(0)$ integrable and that the origin $r=0$ is the inaccessible entrance boundary. The whole aspect of sample behaviors on the interval $(0,\infty)$ is summarized in Fig.7.9. This implies entirely the same result as already described in Section 6.5.

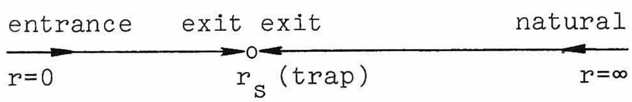


Fig.7.9 The Whole Aspect of Sample Path Behaviors for r(t)-process determined by Eq.(7.62)

7.5 Special Choice of Stabilizing Signals and Unsuccesses of Stabilization

In the previous section, successes of noise stabilization were reported by choosing stabilizing term of biased sinusoidal type or polynomial type. However, at the present stage, although the general rule was established, the choice of stabilizing noise term is still ad hoc. In order to emphasize the stabilizing noise terms adopted here, in this section, examples of unsuccesses are considered, including the effect of nonlinearities exhibited in dynamical systems to the noise stabilization.

7.5.A White Gaussian Noise Stabilization

White Gaussian noise stabilization implies that, in Eq.(6.4), the function $h(x_1,x_2)$ does not depend on the state variables x_1 and x_2. As an example, let $h(x_1,x_2)=a$ in Eq.(6.4), Then, with the help of Eq.(7.60), we have

$$dx_1 = x_2dt \tag{7.74a}$$

$$dx_2 = -\{\omega^2x_1 + \varepsilon(\alpha x_2 + \rho x_1^3)\}dt - \delta a dw. \tag{7.74b}$$

Classification of the boundaries is listed in Table 7.5.

Table 7.5 Classification of the boundaries of the system described by Eq.(7.74)

Interval	σ	μ	$s(r)$	Classification of the boundaries
$r=0$	∞	$<\infty$		inaccessible entrance boundary
$\wr$				
$r=\infty$	∞	∞	$<\infty$	inaccessible natural boundary

From Table 7.5, it may be concluded that the $r(t)$-process which is determined by Eq.(7.74) and originated at a point in the interval $(0,\infty)$ does not arrive at the origin and diverges infinitely with probability one. In general, the white noise stabilization may not be realized.

7.5.B Linearly State-dependent White Noise Stabilization

Letting $h(x_1,x_2)=ax_1$, a similar procedure to Eq.(7.74) brings

(7.75a) $dx_1 = x_2 dt$

(7.75b) $dx_2 = -\{\omega^2 x_1 + \varepsilon(\alpha x_2 + \rho x_1^3)\}dt - \delta a x_1 dw.$

Table 7.6 shows the results of classification of the boundaries.

Table 7.6 Classification of the boundaries of the system described by Eq.(7.75)

Interval	σ	μ	$s(r)$	Classification of the boundaries
$r=0$	∞	∞	∞	inaccessible natural boundary
$\wr$				
$r=\infty$	$<\infty$	$<\infty$		accessible regular boundary

From Table 7.6, it is easily concluded that the r(t)-process determined by Eq.(7.75) diverges infinitely with probability one and the noise stabilization may not be realized.

7.6 Summary

A general rule of noise stabilization for second order nonlinear dynamical systems has been established. On the basis of general consideration of boundary classification, several possible types of stabilization have intuitively been found out. The key notion of approach to realize the noise stabilization is to unify the averaging principle with the Feller's classification criteria of boundaries in which computations of canonical scale and speed measures are required. To realize the stabilization easily, it is the first step to make a trap by a suitable choice of the stabilizing function $h(x_1,x_2)$. However, it is another important aspect to be noted that a selection of stabilizing noise term depends on both the nonlinear system characteristics $g(x_1,x_2)$ and the stabilizing function $h(x_1,x_2)$. From this viewpoint, the general rule developed in Chapter 7 will play an important role to predict the possibility of realizing the noise stabilization.

CHAPTER 8

JUMP PHENOMENON OF NONLINEAR DYNAMICAL SYSTEMS SUBJECTED TO NARROW-BAND RANDOM INPUTS

8.1 Introduction

It is well-known [56] that the jump phenomenon can occur in certain nonlinear dynamical or control systems subjected to sinusoidal inputs, in which, as shown in Fig.8.1, the output amplitude A_0 has a discontinuous jump, as the input amplitude A_I changes. Similarly, in the case where the input to nonlinear systems is stationary random signal, the jump phenomenon occurs between the input variance Ψ_I and the output variance Ψ_0 of the response, as shown in Fig.8.1, in which the response varies as 1→2→3→4→5 as the input variance increases and otherwise changes as 5→4→6→2→1 as the output variance decreases[88]. Accordingly, the dotted line from 3 to 6 does not occur in the practical response. The theoretical approaches up to now were performed based on the describing function method to a sinusoidal input[56] and also the statistical linearization method for stationary random inputs[88]. Furthermore, by these methods, it has been already clarified that the response corresponding to the curve from 3 to 6 in Fig.8.1 is unstable and then does not appear as the real phenomenon[57][89]. However, the approaches described above are first-order approximation methods and then, especially for the stationary random input, we can not give enough explanation of the response by only the fact that, "the response from 3 to 6 is unstable by applying the statistical linearization method." This main reason is based on the fact that, although the statistical linearization method makes possible to evaluate approximately the statistics such as the output variance of the response, it becomes difficult to comprehend stochastically sample path behaviors of the response. Therefore, in order to explicate the generating mechanism of jump phenomenon, it is necessary to obtain other probabilistic informations associated with sample path behaviors. In this chapter, through the evaluation of probability

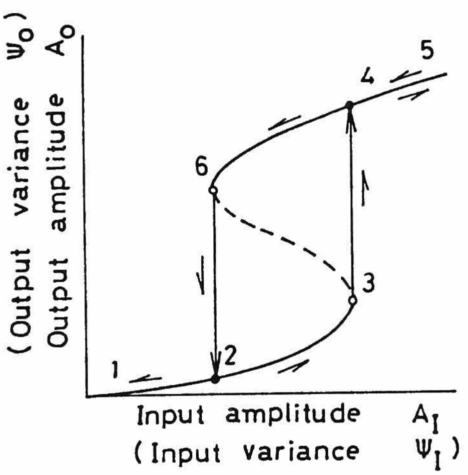


Fig.8.1 Illustration of Jump Phenomenon.

density functions of both the input and the output, we shall clarify the generating mechanism of jump phenomenon occurring in the response of nonlinear dynamical systems. Supposing that the random input to nonlinear systems is a broad-band random signal, it will be possible to evaluate the probabilistic response[58] but, on the other hand, it will need fairly complex calculations to examine the jump phenomenon of sample path behaviors. Then, we shall consider narrow-band random input which signal power is confined to a narrow-band of frequencies. Since such processes are represented as a sinusoidal wave with a randomly varying amplitude and phase, we can proceed the theoretical developments, comparing with the jump phenomenon in nonlinear dynamical system subject to a sinusoidal input which has been already investigated in detail.

In Section 8.2, the stationary probability density of narrow-band random input can be evaluated, which is generated by applying white Gaussian process to lightly damped linear system. In Section 8.3, the stationary response curve can be obtained between the output response of nonlinear systems and the related narrow-band input, through the variational averaging technique. Utilizing this relation between input and output, the stationary probability density function of output response can be derived. Based on the evaluation of the stationary probability density of output response corresponding to the one of input, in Section 8.4, the generating mechanism of jump phenomenon can be verified theoretically by considering the existence area of stationary response. Finally, in Section 8.5, digital simulation studies are demonstrated, showing the validity of the theoretical approach developed here.

8.2 Stationary Probability Density of Narrow-Band Random Input

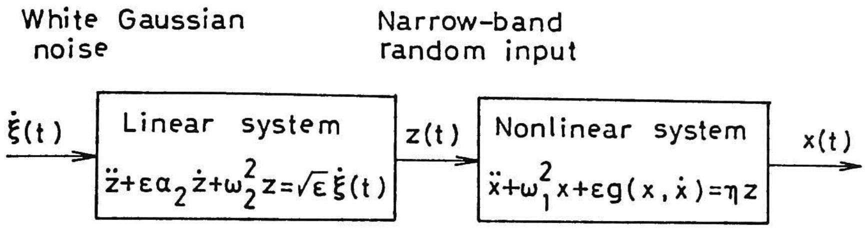


Fig.8.2 Nonlinear Dynamical Systems subjected to Narrow-Band Random Input.

Suppose that the narrow-band random excitation z(t) can be generated as the solution process of the system dynamics,

$$(8.1)\quad \ddot{z} + \varepsilon\alpha_2\dot{z} + \omega_2^2 z = \sqrt{\varepsilon}\,\dot{\xi}(t)$$

where $\dot{\xi}(t)$ is a white Gaussian noise and where "$\cdot$" is differentiation with respect to t, ε and α_2 positive constants and ω_2 natural frequency. Also, α_2 is damping ratio and $\varepsilon\alpha_2 \ll 1$ in order that the spectrum of z(t) will be appreciably narrow-band with central frequency ω_2 for $\dot{\xi}(t)$ with the spectral density $S_{\dot{\xi}}(\omega)=N^2$.

Letting $z=z_1$, $\dot{z}_1=z_2$ in Eq.(8.1), Eq.(8.1) is rewritten by Itô-type stochastic differential equation,

$$(8.2a)\quad dz_1 = z_2 dt$$

$$(8.2b)\quad dz_2 = -\{\varepsilon\alpha_2 z_2 + \omega_2^2 z_1\}dt + \sqrt{\varepsilon}dw(t)$$

where the w(t)-process is the Brownian motion process with the following properties; $E\{dw(t)\}=0$, $E\{(dw(t))^2\}=\sigma^2 dt$, where σ is a constant.

From Eqs.(8.2a) and (8.2b), the following Kolmogorov's backward equation of the z(t)-process is derived:

$$(8.3)\quad \frac{\partial p(t,z_1,z_2)}{\partial t} = z_2\frac{\partial p(t,z_1,z_2)}{\partial z_1} - (\varepsilon\alpha_2 z_2 + \omega_2^2 z_1)\frac{\partial p(t,z_1,z_2)}{\partial z_2} + \frac{\varepsilon}{2}\sigma^2\frac{\partial^2 p(t,z_1,z_2)}{\partial z_2^2}$$

where $p(t,z_1,z_2)$ is the transition probability density function from (z_1,z_2) at time t to (z_{10},z_{20}) at time $t_0(>t)$ for any fixed (z_{10},z_{20}) and t_0. Introducing the new coordinates $z_1 = A\sin(\omega_2 t+\psi_2)$ and $z_2 = \omega_2 A\cos(\omega_2 t+\psi_2)$ in Eq.(8.3), the transformation from the (z_1,z_2)-process to the (A,ψ_2)-process (where A and ψ_2 are random processes) becomes,

$$(8.4)\quad \frac{\partial u}{\partial t} = \varepsilon[-\alpha_2 A\cos^2(\omega_2 t+\psi_2)\frac{\partial u}{\partial A} + \frac{\alpha_2\sin 2(\omega_2 t+\psi_2)}{2}\frac{\partial u}{\partial \psi_2} + \frac{\sigma^2}{2\omega_2^2} \times\{\cos^2(\omega_2 t+\psi_2)\frac{\partial^2 u}{\partial A^2} - \frac{\omega_2\sin 2(\omega_2 t+\psi_2)}{A}\frac{\partial^2 u}{\partial A\partial\psi_2} + \frac{\sin^2(\omega_2 t+\psi_2)}{A^2} \times\frac{\partial^2 u}{\partial \psi_2^2} + \frac{\sin^2(\omega_2 t+\psi_2)}{A}\frac{\partial u}{\partial A} + \frac{\sin 2(\omega_2 t+\psi_2)}{A}\frac{\partial^2 u}{\partial \psi_2^2}\}]$$

where $u=u(t,A,\psi_2)=p\{t,A\sin(\omega_2 t+\psi_2),\omega_2 A\cos(\omega_2 t+\psi_2)\}$.

After somewhat tedious calculations applying averaging principle[46] to Eq.(8.4), the following equation can be obtained[51]:

$$(8.5)\quad \frac{\partial p^*}{\partial t} = \varepsilon[(-\frac{\alpha_2}{2}A + \frac{\sigma^2}{4\omega_2^2 A})\frac{\partial p^*}{\partial A} + \frac{\sigma^2}{4\omega_2^2}\frac{\partial^2 p^*}{\partial A^2} + \frac{\sigma^2}{4\omega_2^2 A^2}\frac{\partial^2 p^*}{\partial \psi_2^2}]$$

where $p^*=p^*(t,A,\psi_2)$. As already mentioned, for a sufficiently small ε, it is known that $p^*(t,A,\psi_2)$ is uniformly approximated as[90]

$$(8.6) \qquad \lim_{\varepsilon\to 0} \sup_t |u(t,A,\psi_2) - p^*(t,A,\psi_2)| = 0.$$

Noting that Eq.(8.5) represents Kolmogorov's backward equation, the system dynamics corresponding to $A(t)$ and $\psi_2(t)$ in Eq.(8.5) are respectively given by

$$(8.7) \qquad dA = \varepsilon\left(-\frac{\alpha_2}{2}A + \frac{\sigma^2}{4\omega_2^2 A}\right)dt + \sqrt{\frac{\varepsilon}{2}}\frac{\sigma}{\omega_2}dw$$

$$(8.8) \qquad d\psi_2 = \sqrt{\frac{\varepsilon}{2}}\frac{\sigma}{\omega_2 A}dw.$$

Now, through the relation that $a(t)=A^2(t)$, the following equation of the $a(t)$-process is derived by applying Itô's differential rule [90] to Eq.(8.7):

$$(8.9) \qquad da = \varepsilon\left(-\alpha_2 a + \frac{1+\sigma^2}{2\omega_2^2}\right)dt + \frac{\sqrt{2\varepsilon a}}{\omega_2}dw$$

where $A=\sqrt{a}$ since A is a positive. Then, the Kolmogorov's forward equation of Eq.(8.9) is

$$(8.10) \qquad \frac{\partial p}{\partial t} = -\varepsilon\frac{\partial}{\partial a}\left(-\alpha_2 a + \frac{\sigma^2+1}{2\omega_2^2}\right)p + \frac{\sigma^2}{2}\frac{\partial^2}{\partial a^2}\left(\frac{2\varepsilon a}{\omega_2^2}p\right).$$

From Eq.(8.10), the stationary probability density function $p^*(a)$ (if there exists,[67]) can be obtained by

$$(8.11) \qquad p^*(a) = \left(\frac{\alpha_2\omega_2^2}{\sigma^2}\right)^{(1+\sigma^2)/2\sigma^2} \Big/ \Gamma\left(\frac{1+\sigma^2}{2\sigma^2}\right) a^{(1-\sigma^2)/2\sigma^2} \times \\ \times \exp\left(-\frac{\alpha_2\omega_2^2}{\sigma^2}a\right).$$

Therefore, from Eq.(8.11), the mean value of square amplitude a is

(8.12) $\quad E\{a\} = (1+\sigma^2)/2\alpha_2\omega_2^2.$

8.3 Stationary Response Characteristics of Nonlinear Systems

As shown in Fig.8.2, we shall explore the stochastic behavior x(t) of second-order nonlinear dynamical systems subject to narrow-band random input z(t) modeled by

(8.13) $$\ddot{x} + \omega_1^2 x + \varepsilon g(x,\dot{x}) = \eta z$$

where g is a nonlinear function, ω_1 natural frequency and ε and η are small positive constants. Equation (8.13) may be considered to be a general expression of mathematical model of nonlinear dynamical systems. However, as it is well-known, it is impossible to treat Eq.(8.13) in the strict sense of mathematics. So, when the solution z(t) of Eq.(8.1) is represented for a sufficiently small ε by

(8.14) $$z(t) = A_1(t)\sin\omega_2 t + A_2(t)\cos\omega_2 t, \quad A^2 = A_1^2 + A_2^2,$$

it is assumed that the solution x(t) of Eq.(8.13) may be approximated by[10]

(8.15) $$x(t) = B_1(t)\sin\nu t + B_2(t)\cos\nu t, \quad B^2 = B_1^2 + B_2^2$$

where the parameter ν is given by

(8.16) $$\nu = \omega_2 - \varepsilon\Delta\nu, \quad \Delta\nu \ll 1 .$$

Now, in order to examine the property of the stationary response x(t) related to narrow-band random input z(t), the variational averaging method [91],[92] is applied to Eq.(8.13) which is described in Appendix A. Substituting Eqs.(8.13) and (8.14) into Eq.(A-10) in Appendix A, it becomes

$$(8.17)\quad \int_{t_i}^{t_i+2\pi/\nu} [-\nu^2 B_1(t)\sin\nu t - \nu^2 B_2(t)\cos\nu t + \omega_1^2\{B_1(t)\sin\nu t$$

$$+ B_2(t)\cos\nu t\} + \varepsilon g(B_1(t)\sin\nu t + B_2(t)\cos\nu t, \nu B_1(t)\cos\nu t$$

$$- \nu B_2(t)\sin\nu t) - \eta(A_1(t)\sin\omega_2 t + A_2(t)\cos\omega_2 t)]$$

$$\times(\delta B_1(t)\sin\nu t - \nu\delta B_2(t)\sin\nu t)dt = 0.$$

However, the calculation of Eq.(8.17) is impossible, because $A_r(t)$ and $B_r(t)$ (r=1 or 2) are unknown parameters. Then, we propose the two-step approximation technique.

First, as already mentioned, the auto-correlation function of $z(t)$-process may be regarded as the harmonic function. Accordingly,

(Proposition-1)For any t with $t_i \le t \le t_i+(2\pi/\nu)$, we let

$$A_r(t) \cong A_{ri}(=\text{constant}), \quad B_r(t) \cong B_{ri}(=\text{constant}).$$

Using Proposition-1, after somewhat tedious calculations, Eq.(8.17) leads to the two algebraic equations :

$$(8.18a)\quad (\omega_1^2 - \nu^2)B_{1i} + I_1(B_{1i}, B_{2i}) = c_1 A_{1i} + c_2 A_{2i}$$

$$(8.18b)\quad (\omega_1^2 - \nu^2)B_{2i} + I_2(B_{1i}, B_{2i}) = c_3 A_{1i} + c_4 A_{2i}$$

where

$$(8.19a)\quad I_1(B_{1i}, B_{2i}) = \frac{\varepsilon\nu}{\pi}\int_{t_i}^{t_i+2\pi/\nu} g(B_{1i}\sin\nu t + B_{2i}\cos\nu t, \nu B_{1i}\cos\nu t$$

$$- \nu B_{2i} \sin\nu t) \sin\nu t dt$$

$$(8.19b) \quad I_2(B_{1i}, B_{2i}) = \frac{\varepsilon\nu}{\pi}\int_{t_i}^{t_i+2\pi/\nu} g(B_{1i}\sin\nu t + B_{2i}\cos\nu t, \nu B_{1i}\cos\nu t$$

$$- \nu B_{2i}\sin\nu t)\cos\nu t dt$$

and also, using Eq.(8.16), from (8.17) and (8.18), it follows that $c_1 = c_4 \cong \eta$ and $c_2 = c_3 \cong 0$.

If the nonlinear function g is given concretely in Eq.(8.19), the second terms of Eqs.(8.18a) and (8.18b) becomes polynomial type with respect to B_{1i} and B_{2i} respectively. Therefore, adding both the square values of Eqs.(8.18a) and (8.18b) respectively and furthermore, using the relations of $a_i = A_i^2 = A_{1i}^2 + A_{2i}^2$ and $b_i = B_i^2 = B_{1i}^2 + B_{2i}^2$, we obtain the following equation,

$$(8.20) \quad a_i = f(b_i) = \sum_{m=1}^{\ell} K_m b_i^m, \quad m=1,2,\cdots,\ell$$

where K_m is a real value. Now, letting be $a_i=\gamma$ (arbitrary positive constant), we consider, from Eq.(8.20), the polynomial equation,

$$(8.21) \quad f(b_i) - a_i = K_\ell b_i^\ell + K_{\ell-1} b_i^{\ell-1} + \cdots + K_1 b_i - \gamma = 0.$$

In order to assume the existence of jump phenomenon (or multi-valued response) between a_i and b_i, the following conditions must be satisfied for Eq.(8.21);

(C-1) $K_\ell > 0$ and $\ell \geq 3$.

(C-2) There exist more than three positive roots. This condition can be satisfied from Descartes Sign Rule[93], if the sign change of the coefficients of Eq.(8.21) is above three times.

(C-3) The solution of $df(b_i)/db_i = 0$ has at least over two positive roots.

In the following, through the relation of Eq.(8.20), we shall examine the stationary probability density $p(b_i)$ of Eq.(8.13). Supposing that the a(t)-process is slowly varying and its period $2\pi/\omega_2$ sufficiently small, the following proposition is introduced for the sample value a_i taken in the every course of period $2\pi/\nu$,

(Proposition-2) $\quad p^*(a) \cong p(a_i).$

Then, from the stationary ergodic hypotheses of the a(t)-process, the following relation holds for a_i,

$$(8.22) \qquad \int_0^\infty a p^*(a) da = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a(t)dt \cong \lim_{N\to\infty} \sum_{i=1}^{N} a_i/N.$$

Furthermore, the following relation holds,

$$(8.23) \qquad \lim_{N\to\infty} \sum_{i=1}^{N} a_i/N \cong \int_0^\infty a_i p(a_i) da_i .$$

As the result, from Proposition-2, the stationary probability density $p(b_i)$ for b_i can be obtained by

$$(8.24) \qquad p(b_i) = p(a_i)\big|_{a_i = f(b_i)} \cdot \frac{df(b_i)}{db_i} .$$

Evaluating the stationary probability density $p(b_i)$ given by Eq.(8.24), we can clarify the stochastic behavior of the stationary response for the system dynamics of Eq.(8.13).

8.4 An Illustrative Example

Based on the general theory established in Section 8.3, we shall consider the nonlinear dynamical system of Duffing-type such that the nonlinear function g in Eq.(8.13) is given by

$$(8.25) \qquad g(x,\dot{x}) = \alpha_1\dot{x} + \beta x^3,$$

which is subjected to narrow-band random signal z(t) of Eq.(8.1). Applying Eq.(8.25) to Eqs.(8.19a) and (8.19b), Eqs.(8.18a) and (8.18b) become,

$$(8.26a) \qquad (\omega_1^2 - \nu^2)B_{1i} - \varepsilon\alpha_1\nu B_{2i} + (3/4)\varepsilon\beta B_{1i}(B_{1i}^2 + B_{2i}^2) = \eta A_{1i}$$

and

$$(8.26b) \qquad (\omega_1^2 - \nu^2)B_{2i} - \varepsilon\alpha_1\nu B_{1i} + (3/4)\varepsilon\beta B_{2i}(B_{1i}^2 + B_{2i}^2) = \eta A_{2i}.$$

On squaring both sides of Eqs.(8.26a) and (8.26b) and calculating the addition of them, we obtain the following result,

$$(8.27) \qquad A_i^2 = c_0 B_i^2\{((3/4)\varepsilon\beta B_i^2 + \omega_1^2 - \nu^2)^2 + (\varepsilon\alpha_1\nu)^2\}$$

where $c_0 \cong 1/\eta^2$. Equation (8.27) can be expressed in the form of Eq.(8.20) by

$$(8.28) \qquad a_i = f(b_i) = c_0\{K_3 b_i^3 + K_2 b_i^2 + K_1 b_i\}$$

where

$$(8.29) \qquad K_3 = (\tfrac{3}{4}\varepsilon\beta)^2, \quad K_2 = \frac{3\varepsilon\beta(\omega_1^2-\nu^2)}{2} \text{ and } K_1 = (\omega_1^2-\nu^2)^2 + (\varepsilon\alpha_1\nu)^2.$$

For the polynomial of Eq.(8.28), we shall examine the conditions for generating jump phenomenon (or multi-valued response) based on the conditions (C-1), (C-2) and (C-3) in Section 8.3.

(i) (C-1) is satisfied since $K_3 > 0$ and $\ell = 3$.

(ii) (C-2) is satisfied, because the change of coefficients is three times such that $K_3 > 0$, $K_2 < 0$, $K_1 > 0$ and $-\gamma < 0$ if $\omega_1 < \nu$.

(iii) If the coefficients of Eq.(8.28) satisfy $K_2 < 0$ and $3K_1K_3 \leq K_2^2 \leq 4K_1K_3$ or equivalently, $\omega_1^2 - \nu^2 + \sqrt{3}\varepsilon\alpha_1\nu < 0$, $df(b_i)/db_i = 0$ has two real positive roots, $b_{id} = (-K_2 - \sqrt{K_2^2 - 3K_1K_3})/3K_3$ and $b_{iu} = (-K_2 + \sqrt{K_2^2 - 3K_1K_3})/3K_3$. Then, it is obvious that (C-3) holds. As the result, the following inequality holds:

(8.30a) $\quad df(b_i)/db_i > 0, \quad 0 < b_i < b_{iu}$ and $b_i > b_{id}$

and

(8.30b) $\quad df(b_i)/db_i \leq 0, \quad b_{iu} \leq b_i \leq b_{id}$.

Then, the response given by Eq.(8.28) represents the multi-valued characteristics as shown in Fig.8.3. The values of a_{id} and a_{iu}

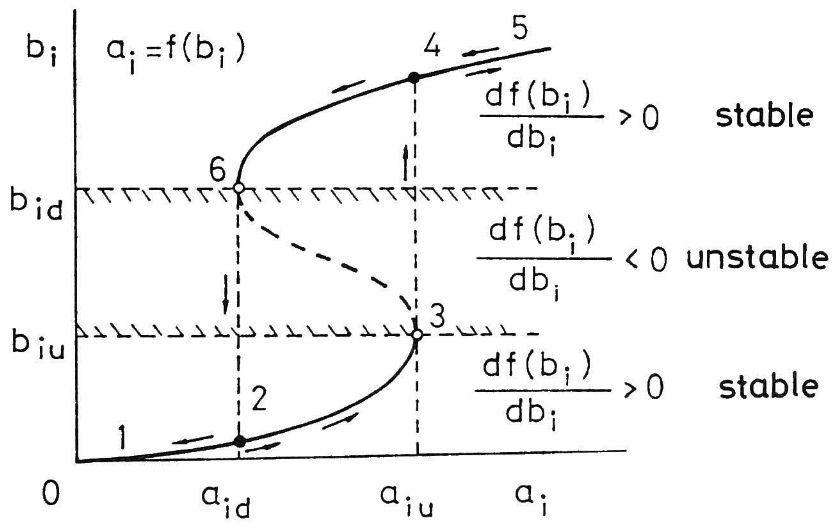


Fig.8.3 Stationary Response Characteristics

corresponding to b_{id} and b_{iu} are given by

$$(8.31a) \qquad a_{id} = (-K_2 - \sqrt{K_2^2 - 3K_1K_3})[-K_2^2 + 6K_1K_3 - K_2\sqrt{K_2^2 - 3K_1K_3}]/27\eta^2K_3^2$$

and

$$(8.31b) \qquad a_{iu} = (-K_2 + \sqrt{K_2^2 - 3K_1K_3})[-K_2^2 + 6K_1K_3 + K_2\sqrt{K_2^2 - 3K_1K_3}]/27\eta^2K_3^2.$$

Nextly, through the relation of Proposition-2, the stationary probability density function of a_i can be obtained from Eq.(8.11) as

$$(8.32) \qquad p(a_i) = \left(\frac{\alpha_2\omega_2^2}{\sigma^2}\right)^{(1+\sigma^2)/2\sigma^2} \Big/ \Gamma\left(\frac{1+\sigma^2}{2\sigma^2}\right) a_i^{(1-\sigma^2)/2\sigma^2} \times$$
$$\times \exp\left(-\frac{\alpha_2\omega_2^2}{\sigma^2} a_i\right) .$$

Accordingly, through the relation of Eq.(8.28), it follows from Eq.(8.24) that the stationary probability density function $p(b_i)$ becomes,

$$(8.33) \qquad p(b_i) = c_1\{f(b_i)\}^{(1-\sigma^2)/2\sigma^2} \exp\left[-\frac{\alpha_2\omega_2^2}{\sigma^2} f(b_i)\right] \frac{df(b_i)}{db_i}$$

where c_1 is a normarized constant. However, the expression of Eq.(8.33) is unreasonable since $p(b_i) < 0$ in the interval $b_{i\ell} \le b_i \le b_{iu}$. On the contrary, it can be proved [94] that the response of b_i is unstable in this interval, as shown in the response curve 3 to 6 of Fig.8.3 (referring to Appendix B). Then, the stationary probability density function $p(b_i)$ becomes,

$$(8.34) \qquad p(b_i) = 0, \qquad b_{i\ell} \le b_i \le b_{iu}.$$

Consequently, the probability density function $p(b_i)$ of Eq.(8.33) can be rewritten in the following modified form,

$$(8.35a) \qquad p(b_i) = c_1'\{f(b_i)\}^{(1-\sigma^2)/2\sigma^2} \exp[-\frac{\alpha_2\omega_2^2}{\sigma^2} f(b_i)] \frac{df(b_i)}{db_i}$$

$$b_i < b_{iu} \text{ and } b_{id} < b_i$$

$$(8.35b) \qquad p(b_i) = 0 \qquad b_{id} \le b_i \le b_{iu}$$

where c_1' is a normarized constant. From Eq.(8.35), the jump phenomenon occurs at the both points of (a_{iu}, b_{iu}) and (a_{id}, b_{id}) in Fig.8.3 according to the transition of $p(a_i)$ given by Eq.(8.32), based on the change of $E\{a_i\}$. In other words, since the point 3 is unstable one and otherwise the point 4 stable one in Fig.8.3, the jump of magnitude b_i occurs from 3 to 4. Furthermore, the similar jump phenomenon occurs from the unstable point 6 to the stable point 2.

Here, we shall explain the occurrence of the phenomenon described above in detail using the stationary probability density functions $p(a_i)$ and $p(b_i)$. Now, the system parameters of Eqs.(8.1) and (8.13) are set as

(8.36a) $\varepsilon = 0.01$, $\omega_2 = 5.276$, $\alpha_2 = 0.1$, $\omega_1 = 5$, $\eta = 1$ and $\nu = 5.27$,

where parameters of the nonlinear function g in Eq.(8.25) are given by

(8.36b) $\alpha_1 = 20$ and $\beta = 500$

and also the variance of the white Gaussian noise $\dot{\xi}(t)$ is $\sigma^2 = 0.1$.

In this case, the stationary response curve of Eq.(8.28) was obtained as shown in Fig.8.4. Figure 8.5 represents the stationary probability density functions $p(a_i)$ of Eq.(8.32) which are shifted from A-1 to A-6 successively by changing the initial condition a_0. Here, the initial values a_0 are set as 0, 0.3, 0.59, 0.8, 0.95 and

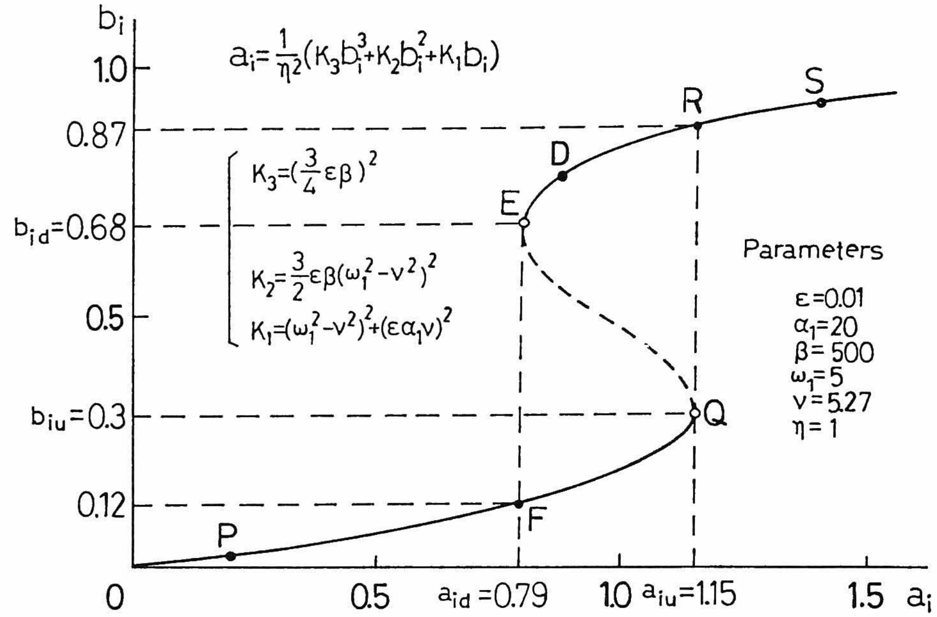


Fig.8.4 Stationary Response Curve of Eq.(8.28).

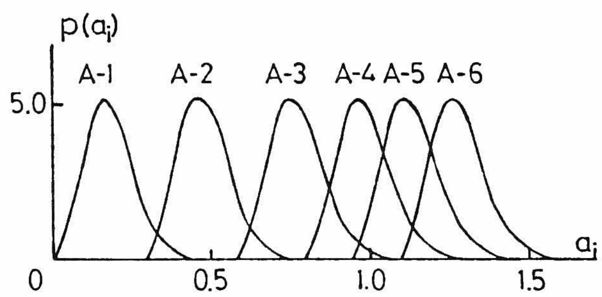


Fig.8.5 Transition of $p(a_i)$ with the Change of Initial Conditions.

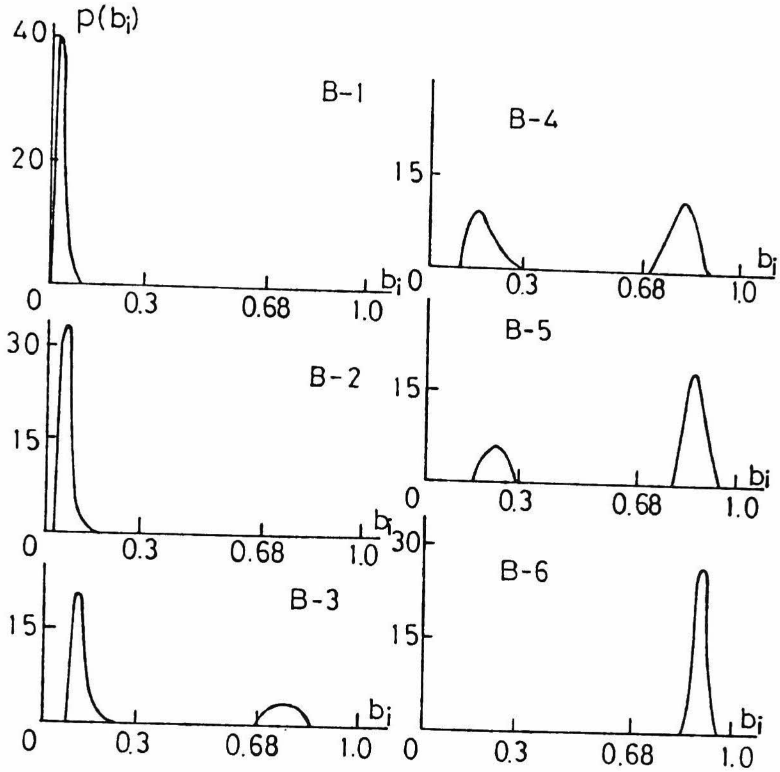


Fig.8.6 Transition of $p(b_i)$ corresponding to the Transition of $p(a_i)$.

1.1. The figures of the transition of $p(b_i)$ were obtained, through the calculations of Eq.(8.35) as B-1 to B-6 in Fig.8.6 respectively corresponding to A-1 to A-6 in Fig.8.5. In Fig.8.6, it should be noted that the interval $0.3 \leq b_i \leq 0.68$ shows an unstable region.

Now, we shall consider the case where $p(a_i)$ increases in the sense of mean value by changing the initial condition a_0. When $p(a_i)$ is shifed to A-1, A-2 and A-3 successively, the $b_i(t)$-process shows the stationary response corresponding to the curve $\widehat{PQ}$ in

Fig.8.4 and then the jump from the point Q to R does not occur practically, as known by $p(b_i)$ of B-1 to B-3 in Fig.8.6. However, as shown by $p(b_i)$ of B-4 and B-5 when shifted to A-4 and A-5, it can be possible that there exists the stationary response corresponding to the curve $\widehat{ES}$ in Fig.8.4. This implies that the stationary response corresponding to $\widehat{PQ}$ jumps at the point Q and shifts to the stationary states $\widehat{ES}$. Furthermore, when $p(a_i)$ shifts to A-6, there exists only the stationary response to the curve $\widehat{ES}$ and a jump does not occur.

On the other hand, the similar phenomenon can be illustrated in the case that $p(a_i)$ decreases in the sense of mean value of a_i. That is, we shall consider the case where $p(a_i)$ moves from A-6 to A-1 in Fig.8.5 successively. Then, from $p(b_i)$ with the transition of B-4 and B-3 corresponding to A-4 and A-3, it follows that the stationary response to the curve $\widehat{ES}$ jumps at the point E and shifts to the stationary state to $\widehat{PQ}$.

8.5 Digital Simulation Studies

In this section, the experiments can be performed to verify the occurrence of jump phenomenon in the nonlinear stochastic systems by observing the sample path behaviors. Letting be $z=z_1$, $\dot{z}_1=z_2$, $x=x_1$ and $\dot{x}_1=x_2$, Eqs.(8.1) and (8.13) can be transformed as

(8.37a) $$dz_1/dt = z_2,$$

(8.37b) $$dz_2/dt = -\varepsilon\alpha_2 z_2 - \omega_2^2 z_1 + \sqrt{\varepsilon}\dot{\xi}(t)$$

and

(8.38a) $$dx_1/dt = x_2,$$

(8.38b) $\quad dx_2/dt = -\varepsilon\alpha_1 x_2 - \omega_1^2 x_1 - \varepsilon\beta x_1^3 + \eta z.$

Numerical calculations were performed by Runge-Kutta-Gill Method[95] to obtain the values $z_{r,n}$ and $x_{r,n}$ at time t_n from the values $z_{r,n-1}$ and $x_{r,n-1}$ at time t_{n-1} where r=1 or 2 and n=1,2,⋯. Using the sample values $z_{1,n}$, $z_{2,n}$, $x_{1,n}$ and $x_{2,n}$ obtained above, $a(t_n)$ and $b(t_n)$ can be computed respectively as

(8.39) $\quad a(t_n) = z_{1,n}^2 + (z_{2,n}/\omega_2)^2$

and

(8.40) $\quad b(t_n) = x_{1,n}^2 + (x_{2,n}/\nu)^2.$

In these experiments, the parameters of Eqs.(8.37) and (8.38) were set as (8.36). Also, the variance of $\dot{\xi}(t)$ were given by $\sigma^2 = 0.1$ and 0.8.

First, the solution process z(t) of Eq.(8.37) has the properties such that the spectral density $S_z(\omega)$ and the auto-correlation function $R_z(\tau)$ become respectively

(8.41) $\quad S_z(\omega) = 0.1\sigma^2/\{(\omega^2 - 5.276^2)^2 + (0.001\omega)^2\}$

and

(8.42) $\quad R_z(\tau) = E\{A(t)A(t+\tau)\}(\cos 5.276\tau)/2,$

which are shown in Figs.8.7 and 8.8. Furthermore, from Eq.(8.8), the stationary probability density function $p(a_i)$ is obtained as shown in Fig.8.9.

Nextly, the stationary response characteristics was already

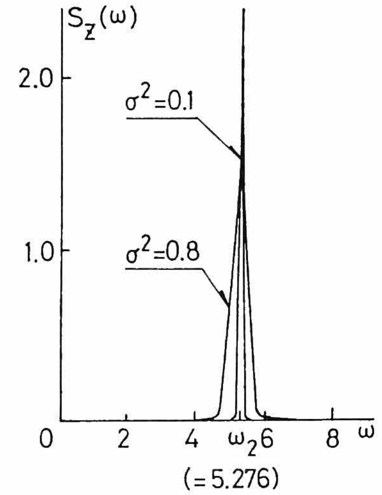


Fig.8.7 Spectral Density of the z(t)-process

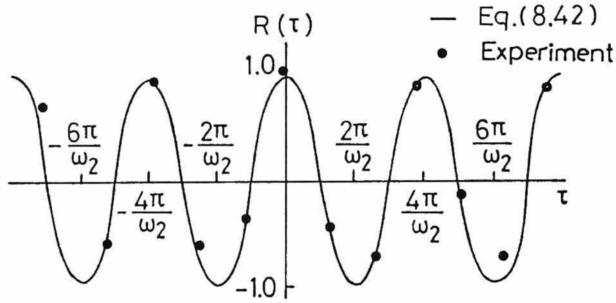


Fig.8.8 Auto-correlation function of the z(t)-process

obtained in Fig.8.4 as the relationship between the input and output amplitudes. Thus, along the line of the response curve obtained above, the main problem is to examine the occurrence of jump phenomenon at the points Q or E for the $b_i(t)$-processes in the case where the $a_i(t)$-processes increase or decrease in the sense of mean value $E\{a_i\}$. By the way, it is sufficient if each one data of

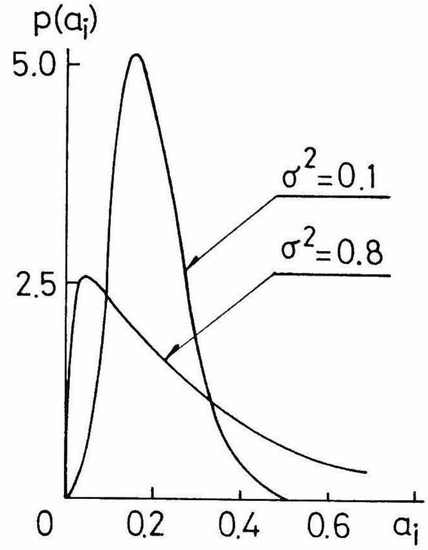


Fig.8.9 Stationary Probability Density Function of z(t).

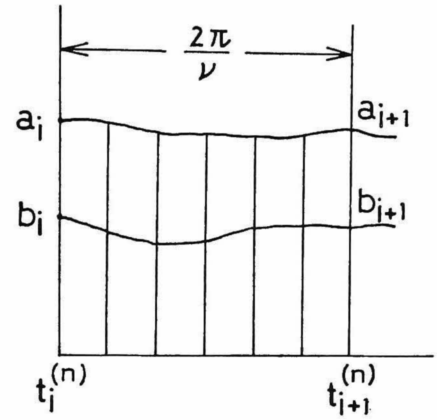
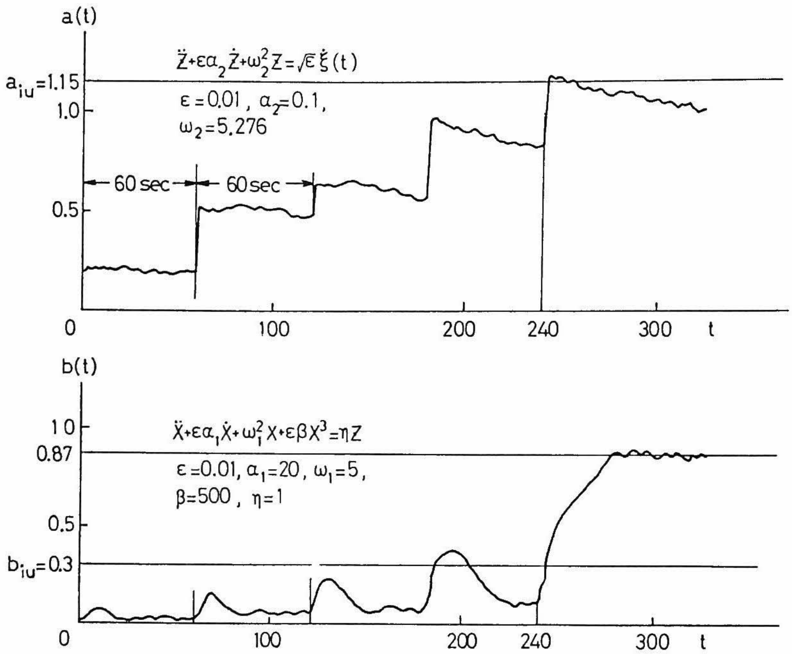


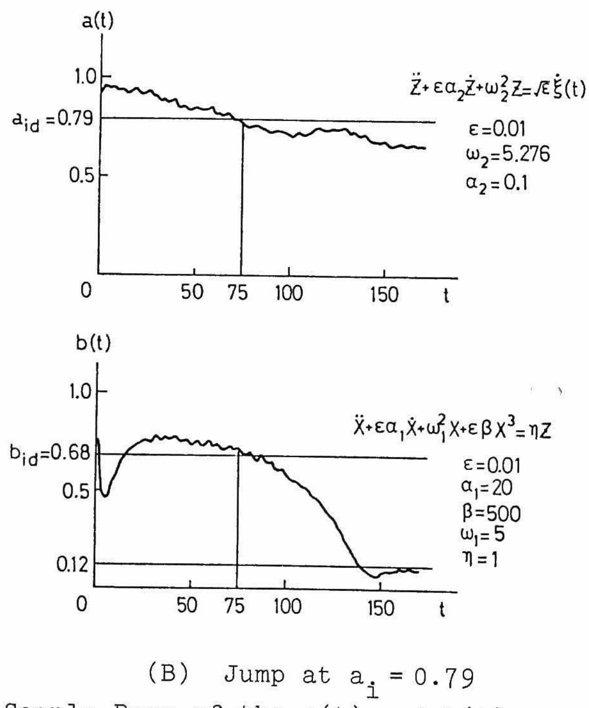
Fig.8.10 Sample Values a_i and b_i.

sample values a_i and b_i is taken in the each interval $t_i \le t \le t_i + 2\pi/\nu$. However, in this simulation, we confirmed that the Proposition-1 holds by picking up and comparing with six data in the interval $t_i \le t \le t_i + 2\pi/\nu$, as shown in Fig.8.10. Here, $a_i = a(t_n^i)$ and $b_i = b(t_n^i)$ and also $t_i = t_i^{(n)}$ represents $t_i^{(n)} = t_n$, $n = 6i-5, 6i-4, \cdots, 6i$. In all experiments, the time interval of sample values was set as $\delta = t_{n+1} - t_n = 0.2$.

The results of digital simulation studies are shown in Figs. 8.11 and 8.12. Figure 8.11 shows the sample runs of the a(t) and b(t) processes in the case of $\sigma^2 = 0.1$, in which (A) is the case where the a(t)-process increases and where (B) the case that the a(t)-process decreases. In Fig.8.11(A), the increase of the a(t)-proc-

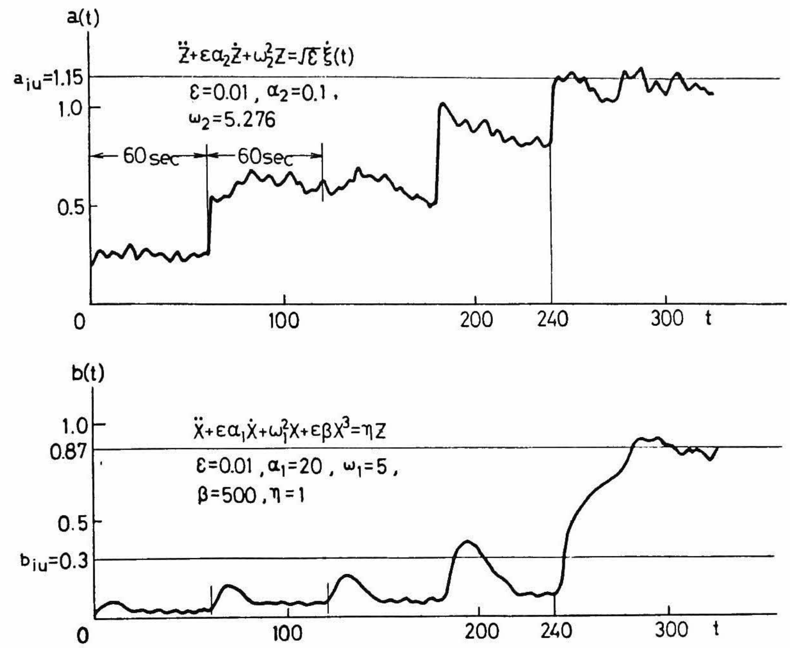


(A) Jump at $a_i = 1.15$.

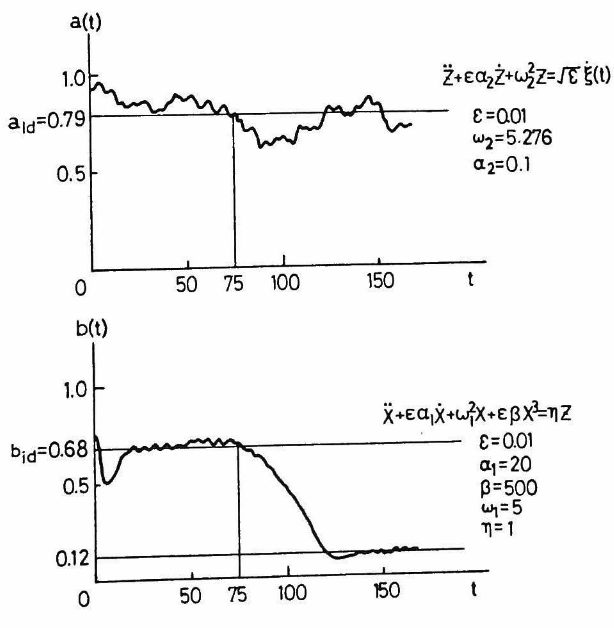


(B) Jump at $a_i = 0.79$

Fig.8.11 Sample Runs of the a(t) and b(t) processes (σ^2=0.1).



(A) Jump at $a_i = 1.15$



(B) Jump at $a_i = 0.79$

Fig. 8.12 Sample Runs of the a(t) and b(t) processes ($\sigma^2 = 0.8$)

ess was performed by changing the initial value in every 60 sec. And the first initial value was set at the point $P = (a(0), b(0)) = (0.2, 0.02)$ in Fig.8.4. In the case where the magnitude of the a(t)-process is smaller than $a_{iu} = 1.15$ (in the time interval $0 \leq t \leq 240$), the b(t)-process increases gradually along the stationary response curve $P \rightarrow F \rightarrow Q$ in Fig.8.4. Once the a(t)-process is over $a_{iu} = 1.15$ $(t \geq 240)$, the b(t)-process varies abruptly from the point Q to R in Fig.8.4 and then yields the jump to the value of $b = 0.87$.

Afterwards, the b(t)-process converges to the response curve $\widehat{ES}$ in Fig.8.4. In Fig.8.11(A), the overshoots of the b(t)-process show the transient phenomenon at each initial time. On the other hand, Fig.8.11(B) shows the sample run of the b(t)-process in which the a(t)-process is slowly decreasing. The initial value was set at the point $D = (0.93, 0.75)$ in Fig.8.4. When the sample values of the a(t)-process are over the value of $a_{id} = 0.79$ (in the time interval $0 \leq t \leq 75$), the b(t)-process moves along the response curve $\widehat{DE}$ in Fig.8.4. However, once the a(t)-process is smaller than the value of $a_{id} = 0.79$ at time $t \cong 75$, the b(t)-process changes suddenly from the point E to F and then occurs the jump to the value of $b = 0.12$. After the time $t = 140$, the b(t)-process converges to the response curve $\widehat{FP}$ in Fig.8.4. Figure 8.12 shows the simulation results in the case of $\sigma^2 = 0.8$. As well as the case of Fig.8.11, it can be observed that there exist jump phenomena at the points $a_{iu} = 1.15$ in (A) and $a_{id} = 0.79$ in (B). From the experimental results of Fig.8.11 and 8.12, it was verified that there exist the jump phenomena as the practical behaviors, which was already clarified theoretically. Furthermore, in the interval $a_{id} \leq a_i \leq a_{iu}$, it was ascertained that the response in the interval $\widehat{EQ}$ shows unstable behaviors.

8.6 Summary

The past analytical studies of jump phenomenon in the stochastic systems had been developed by evaluating approximately the relationship of variances between the input and the output amplitudes[90]. In this section, we shall consider the analytical results mentioned here from this viewpoint of the variance evaluation. First, let

$$(8.43a)\qquad \bar{a} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} a(t)dt$$

and

$$(8.43b)\qquad \bar{b} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} b(t)dt.$$

The relation between $\bar{a}$ and $\bar{b}$ can be obtained as follows by replacing a_i and b_i in Eq.(8.28) with $\bar{a}$ and $\bar{b}$,

$$(8.44)\qquad \bar{a} = f(\bar{b})$$

$$= [(\tfrac{3}{4}\varepsilon\beta)^2\bar{b}^3 + \tfrac{3}{2}\varepsilon\beta(\omega_1^2 - \nu^2)\bar{b}^2 + \{(\omega_1^2 - \nu^2) + (\varepsilon\alpha_1\nu)^2\}\bar{b}]/\eta^2.$$

The relationship of Eq.(8.44) is shown in the curve I of Fig.8.13. On the other hand, using both Eqs.(8.32) and (8.33), we shall examine the relation between $\bar{a}_i=E\{a_i\}$ and $\bar{b}_i=E\{b_i\}$. Referring to the characteristics of the stationary response in Fig.8.4, noticing that the interval of $0.3 \leq b_i \leq 0.68$ is the unstable region and $p(b_i) = 0$ in this interval, the $\bar{b}_i$ can be obtained by

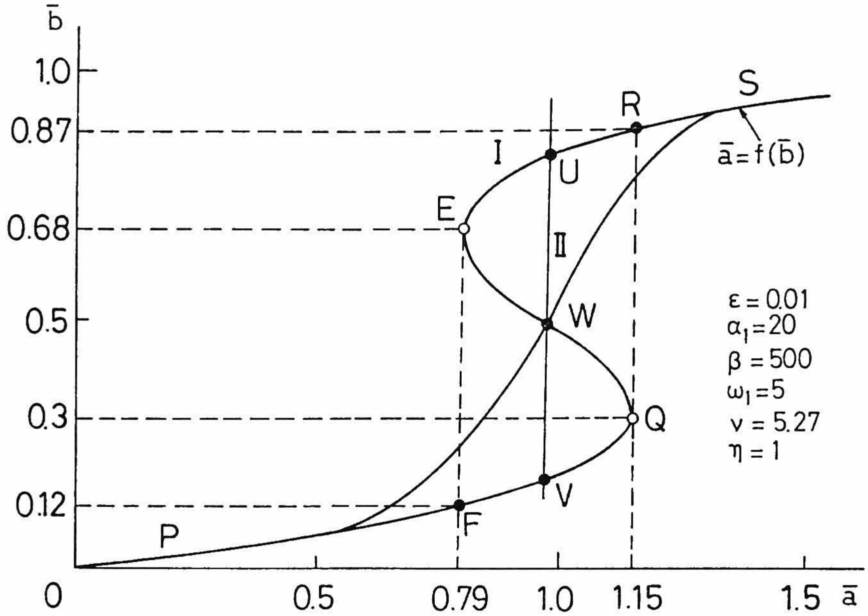


Fig.8.13 Evaluation of the Stationary Response by the Mean Value.

$$(8.45)\qquad \bar{b}_i = \int_0^{\infty} b_i p(b_i)db_i = \int_0^{0.3} b_i p(b_i)db_i + \int_{0.68}^{\infty} b_i p(b_i)db_i .$$

From the stationary hypothesis of the a_i process described in Proposition-2 and furthermore the stationary hypothesis of the b_i process, it can be regarded as $\bar{a}_i = \bar{a}$ and $\bar{b}_i = \bar{b}$. Accordingly, the calculation result of Eq.(8.45) yields the response curve II in Fig.8.13. Since, obviously from Eq.(8.45), the curve II was calculated by excluding the unstable region, it can be considered that the value W of the curve II is the average of the sample values U and V. Then, from this stationary response curve II , the existence of jump phenomenon can not be verified.

CHAPTER 9

DISCUSSION AND CONCLUSIONS

A comprehensive study of the dynamics of the nonlinear stochastic differential equations has been performed and applied to the second order nonlinear dynamical systems subjected to random inputs. The results and conclusions of this study will be discussed in this section.

In Chapter 3 and 4, three new approaches has been developed to examine the existence of stationary probability density functions for nonlinear dynamical system response. The first was concerned with a choice of a suitable Lyapunov-like function, the second was concerned with the application of an arbitrary martingale function and the third was concerned with the construction of conditions for some stochastic model based on the classification criteria of the boundaries. These three approaches presented here gave sufficient conditions for guaranteeing the existence of the stationary solution of the Fokker-Planck equation. The first two approaches are applied to nonlinear stochastic systems with singular points and the third is suitable for nonlinear systems with no singular points. Naturally, a choice of the type of Lyapunov-like functions depends on the nonlinear characteristics contained in the dynamical system considered. However, the present methods provide a new light in the exploration of the asymptotic behavior of a wide class of nonlinear dynamical systems around singular points.

In Chapter 5, a realizable approach was presented to analyze the asymptotic stability of a general class of nonlinear stochastic dynamical systems. The basic notion presented here was a choice of the stochastic Lyapunov function with an advantage that influences of initial values of the system states came out. Introducing the concept of random evolution [35][36], the stability analysis was extended to a general class of nonlinear dynamical systems involving two kinds of random parameters modeled by a white Gaussian and a Markov chain processes respectively. Throughout this study, the relation between the asymptotic behavior of nonlinear stochastic systems and the domain of their initial values are examined by using the useful theorems giving sufficient conditions for the asymptotic stability with the probability appraisal.

In Chapters 6 and 7, a method of noise stabilization for the second order nonlinear dynamical systems has been developed. On the basis of stability criteria established, two possible types of noise terms have, intuitively, been found out ; one is a biased sinusoidal signal and another a biased polynomial type signal. The principal line of attack to realize the noise stabilization was to unify the averaging principle with the Feller's classification criteria of boundaries in which the computation of canonical scale and speed measures was required. To realize the stabilization easily, it was the first step to make a trap by a suitable choice of the stabilizing function $h(x,\dot{x})$. However, it is important aspect to be noted that a selection of stabilizing noise term depends on both the nonlinear system characteristics $G(x,\dot{x})$ and the stabilizing function $h(x,\dot{x})$. From this viewpoint, the general rule for noise stabilization of unstable nonlinear dynamical systems was established. The general rule developed in Chapter 7 will play an important role to predict the possibility of realizing the noise stabilization.

In Chapter 8, a new approach has been developed to analyze the jump phenomenon occurring in the nonlinear dynamical systems subjected to a narrow-band random input. Throughout the variational averaging technique, the multi-valued response curve was obtained approximately in the stationary state. Based on the relation (8.28), the stationary probability density function of the output amplitude was derived, by which the generating mechanism of jump phenomenon was clarified. The main result of this study is that the analysis of jump phenomenon has been performed by evaluating the probability density function of the input and output amplitudes, in order to examine sample path behaviors of jump phenomenon. It was shown that, through digital simulation studies, the occurrence of jump phenomenon was obviously explained by the observation of sample path behaviors.

APPENDICES

Appendix A Variational Averaging Method [91],[92]

The variational averaging method is based on the principle of virtual work which may be expressed in the form of Hamilton's Modified Principle,

$$\text{(A-1)} \qquad \int_{t_1}^{t_2} \sum_{j=1}^{n} \left[-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \frac{\partial L}{\partial q_j} + Q_j(t) \right] \delta q_j \, dt = 0, \quad j=1,\cdots,n$$

where L is the Lagrangean function, the q_j are generalized coordinates and $Q_j(t)$ are generalized forces given by random input processes. Since the δq_j are independent variations in the coordinates, the term in the brackets must be equal to zero individually. This normally results in the set of second order differential equations,

$$\text{(A-2)} \qquad G_j[\, q_1,\cdots,q_n \,;\cdots;\, \ddot{q}_1,\cdots,\ddot{q}_n \,;\, Q_j(t) \,] = 0, \quad j=1,\cdots,n$$

where $G_j[\cdot]$ represents the dynamics describing the system.

It is assumed that $q_j(t)$ in Eq.(A-2) are approximated by

$$\text{(A-3)} \qquad q_j(t) \cong \hat{q}_j(t) = \hat{q}_j(B_{j1}(t),\cdots,B_{j\ell}(t)\,;\,t).$$

In Eq.(A-3), the function $\hat{q}_j(t)$ is chosen to be a known function of time and the parameters $B_{j\ell}(t)$ are random processes which are to be determined in such a way as to make the assumed solution a best fit to the exact solution. We must explore a criteria for the selection of the $B_{j\ell}(t)$ so that the approximated solution is as good as possible. The criteria selected here is that Eq.(A-1) is satisfied over a specified time interval of interest $t \epsilon [t_1, t_2]$. We then require that

$$\text{(A-4)} \qquad \int_{t_1}^{t_2} G_j[\hat{q}_1, \cdots, \ddot{\hat{q}}_n ; Q_j(t)] \delta\hat{q}_j \, dt = 0 , \quad j=1,\cdots,n.$$

Since

$$\text{(A-5)} \qquad \delta\hat{q}_j = \sum_{j=1}^{\ell} \frac{\partial \hat{q}_j(B_{j\ell})}{\partial B_{j\ell}} \delta B_{j\ell} ,$$

and $B_{j\ell}(t)$ may be varied independently, we obtain the following variational equation from Eq.(A-4),

$$\text{(A-6)} \qquad \int_{t_1}^{t_2} G_j(t) \sum_{j=1}^{\ell} \frac{\partial \hat{q}_j(B_{j\ell})}{\partial B_{j\ell}} \delta B_{j\ell} \, dt = 0$$

where $G_j(t) = G_j[\hat{q}_1, \cdots, \ddot{\hat{q}}_n ; Q_j(t)]$.

Now, considering the system equation (8.13), the $\hat{q}_j$, $Q_j(t)$ and $G_j(t)$ mentioned above are given as follows:

$$\text{(A-7)} \qquad \hat{q}_1(B_1, B_2) = \hat{x}(t) = B_1(t)\sin\nu t + B_2(t)\cos\nu t,$$

$$\text{(A-8)} \qquad Q_1(t) = \hat{z}(t) = A_1(t)\sin\omega_2 t + A_2(t)\cos\omega_2 t$$

and

$$\text{(A-9)} \qquad G_1(t) = \ddot{\hat{x}} + \omega_1^2\hat{x} + \varepsilon g(\hat{x}, \dot{\hat{x}}) - \eta\hat{z}(t),$$

where $\ell=2$ and $n=1$. Therefore, letting $t_1 = t_i$, $t_2 = t_i + 2\pi/\nu$, Eq.(A-6) is rewritten by

$$\text{(A-10)} \qquad \int_{t_i}^{t_i+2\pi/\nu} [\ddot{\hat{x}} + \omega_1^2\hat{x} + \varepsilon g(\hat{x}, \dot{\hat{x}}) - \eta\hat{z}]\{\frac{\partial \hat{z}}{\partial B_1}\delta B_1 + \frac{\partial \hat{z}}{\partial B_2}\delta B_2\} dt = 0$$

and then Eq.(8.17) can be obtained.

Appendix B Stability Analysis of Stochastic Response[10]

Introducing the rotating coordinates, the solution process of the system (8.13) is represented by

(B-1) $x(t) = B(t)\sin\phi_1(t), \quad \dot{x}(t) = \nu B(t)\cos\phi_1(t)$

where $\phi_1(t) = \nu t + \psi_1(t)$. In Eq.(B-1), the random variable B(t) satisfies the relation (8.27) in $t\epsilon[t_i, t_i + 2\pi/\nu]$. From (B-1), it is easily shown that

(B-2) $B^2 = x^2 + \dot{x}^2/\nu^2, \quad \psi_1 = \tan^{-1}(\nu x/\dot{x}) - \nu t.$

Letting $b = B^2$ and differentiating with respect to time, it becomes that

(B-3) $\dot{b} = 2\dot{x}(\nu^2 x + \ddot{x})/\nu^2, \quad \dot{\psi}_1 = -x(\nu^2 x + \ddot{x})/\nu b.$

Substituting Eqs.(8.1), (8.13) and (B-1) into (B-3) and deleting the terms of higher harmonics, the following truncated equations are found to be

(B-4a) $\dot{b} = H_1[b, \psi_1] = -\epsilon\alpha_1 b + \frac{\eta\sqrt{a \cdot b}}{\nu}\sin(\psi_2 - \psi_1)$,

(B-4b) $\dot{\psi}_1 = H_2[b, \psi_1] = \frac{\omega_1^2 - \nu^2}{2\nu} + \frac{3\epsilon\beta}{8\nu}b - \frac{\eta}{2\nu}\sqrt{\frac{a}{b}}\cos(\psi_2 - \psi_1).$

Since the fluctuation x(t) of the system (8.13) is the narrow-band random process, the amplitude b and the phase ψ_1 of the response process manage to take the quasi-stable values causing that $\dot{b}$ and $\dot{\psi}_1$ in Eqs.(B-4a) and (B-4b) make zero. Let δb and $\delta\psi_1$ be the small deviations of b and ψ_1 from their stable values. Then, using Eqs.(B-4a) and (B-4b), the following linearized equations for δb and $\delta\psi_1$ are obtained by

(B-5) $\quad \delta\dot{b} = a_{11}\delta b + a_{12}\delta\psi_1, \quad \delta\dot{\psi}_1 = a_{21}\delta b + a_{22}\delta\psi_1,$

where

(B-6) $$a_{11} = \frac{\partial H_1[b,\psi_1]}{\partial b} = -\frac{\varepsilon\alpha_1}{2}, \quad a_{12} = \frac{\partial H_1[b,\psi_1]}{\partial\psi_1} = -\frac{b}{\nu}\{\omega_1^2-\nu^2+\tfrac{3}{4}\varepsilon\beta b\},$$

$$a_{21} = \frac{\partial H_2[b,\psi_1]}{\partial b} = \frac{1}{4\nu}\{\tfrac{9}{4}\varepsilon\beta + \frac{\omega_1^2-\nu^2}{b}\} \text{ and } a_{22} = \frac{\partial H_2[b,\psi_1]}{\partial\psi_1} = -\frac{\varepsilon\alpha_1}{2}.$$

Then, the characteristic equation is

(B-7) $$\lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0.$$

By Routh-Hurwitz method, it is easily seen that necessary and sufficient conditions for the stability of Eq.(B-5) are obtained as

(B-8a) $$a_{11} + a_{22} = -\varepsilon\alpha_1 < 0$$

(B-8b) $$a_{11}a_{22} - a_{12}a_{21} = \frac{1}{(2\nu)^2}[3(\tfrac{3}{4}\varepsilon\beta)^2 b^2 + 2\{\tfrac{3}{2}\varepsilon\beta(\omega_1^2 - \nu^2)\}b + (\varepsilon\nu\alpha_1)^2 + (\omega_1^2 - \nu^2)^2] > 0.$$

Note that the first condition is always fulfilled for $\alpha_1 > 0$. The second condition can be expressed by

(B-9) $$\frac{1}{c_0(2\nu)^2}\frac{df(b_i)}{db_i} > 0$$

where $f(b_i)$ is given by Eq.(8.28). Therefore, the stability of the amplitude b can be evaluated by examining the sign of $df(b_i)/db_i$. The following result can be obtained with respect to the stationary response curve of Eq.(8.28) in Fig.8.3 ; the amplitude responses on the branches $\widehat{23}$ and $\widehat{64}$ in Fig.8.3 correspond to stable values of b, since the derivative of $f(b_i)$ with respect to the amplitude b_i is positive, whereas the response on the branch $\widehat{36}$ is unstable because the derivative of $f(b_i)$ is negative. Then, a jump of the output amplitude b occurs from the point 3 to the point 4, because the point 3 is unstable and the point 4 is stable. Similarly, a jump of the output amplitude occurs from the unstable point 6 to the stable point 2.

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